1D Ising Model

The theory of domain walls can be directly used to describe the low temperature thermodynamics of a one dimensional (1D) Ising model, and demonstrate the lack of any finite temperature ordering in this low dimensional limit.

In the previous lecture we examined the structure of a single flat domain wall. In reality, droplets and the domain walls describing their surface can assume complicated shapes, rendering our model calculation of limited applicability and relevance. The situation is different in one dimension. Here the domain wall is a "point" object, and the theory in its original form is sufficient to describe all the essential physics.

Let us consider an Ising ferromagnet at $T = 0$. In the ground state all the spins are aligned in a given (call it "up") direction. At low but finite temperature, we can imagine that "droplets" form, consisting of "down" spins. The "surface" of such a droplet consists of two domain walls, call then a "kink" and an "anti-kink". A kink and an anti-kink can "annihilate" each other when they are brought into contact - the droplet is then destroyed. But how does the energy of the system change when we modify the distance between the kink and the anti-kink? Well - not at all! In general, creating each domain wall costs a certain energy per unit area (its surface tension). In one dimension, though, the area of each domain wall is "one", and thus each the energy of each domain wall is a finite number.

Landau's argument for absence of ordering in 1D

In order to have an ordered phase, most spins have to point in one direction. To see how stable this ordering is, we discuss the energetics of introducing domain walls in the ground state, creating droplets, thus reducing the magnetization. What is a free energy of creating a droplet? Well, its energy is, as we have seen, that of creating two domain walls

$$E_{drop} = 2\varepsilon_{dw} \sim J.$$

What about the entropy? Well it is determined by the number of accessible states at that energy. Landau noted that the number of configurations for a droplet proportional L^2, where L is the system size. This is true, since we can put one domain wall in N different places,

and the other one also in L different places (up to a numerical prefactor), since moving the domain walls **costs us no energy** for models with **short-range** interactions. This the entropy is

$$S_{drop} \sim \ln(L) \to \infty$$

in the thermodynamic limit. Since the free energy $F = E - TS$, the entropy will always win for a large system, and the free energy will always be lowered by introducing domain walls in the system, **no matter how low the temperature is!** Therefore, we'll have a finite density of domain walls present in the system at any finite temperature. Note that moving apart the two domain walls of a down-spin droplet reduces the overall magnetization. Since this costs us no energy, the domain walls will be randomly positioned, hence any given spin has equal likelihood to point up or down. The ordered phase is thus destroyed at any finite temperature.

Note that the same argument does not work in $d > 1$!! This is true, since in d dimensions, the droplet surface are scales as L^{d-1}. To destroy long-range order, we have to put a domain wall through the entire system, we have to pay an energy that grows much fastyer then the system size, then the entropy we gain (which scales as $\ln L$ in all dimensions).

Low T thermodynamics

But what happens as we lower the temperature? The answer is simple: we have fewer and fewer domain walls, and their separation grows. We get longer and longer regions that have aligned spins - the correlation length grows! But how does it precisely depend on temperature? To find out we must calculate the partition function of the system. This is easily done, since each spin configuration can be represented as a superposition of a given number of kinks and anti-kinks, each of which costs exactly energy ε_{dw} to be created.

Thus the energy of each configuration can be determined by simply counting the number of domain walls in the system! Thus, the partition function is that corresponding to a gas of noninteracting particles moving in one dimension, with chemical potential $\mu = \varepsilon_{dw}$. Each site can be either occupied or empty, hence the partition function is

$$Z = \left(1 + e^{-\mu/T}\right)^L,$$

where L is the number of unit cells in the system

The average number of particle is obtained as

$$\langle N\rangle = -\frac{\partial}{\partial\left(\beta\mu\right)}\ln Z = L\frac{e^{-\mu/T}}{1+e^{-\mu/T}} \approx Le^{-\mu/T},$$

at low temperature. The correlation length can be estimated as the average distance between domain walls, which is inversely proportional to their density

$$\xi = \frac{L}{\langle N\rangle} \approx e^{\mu/T}.$$

To convince us that this indeed is a correlation length, we can try to compute a correlation function $\chi(R) = \langle S(R)S(0)\rangle$ for a spin configuration with a random distribution of domain walls (at low density they form a noninteracting gas) with an average density $\langle N\rangle / L = \xi$. A simple calculation (homework problem HW#7.1) shows that indeed $\chi(R) \sim \exp\{-R/\xi\}$, so that ξ indeed is the properly defined (true) correlation length in a system with domain walls.

The true correlation length ξ is very different then the mean-field correlation length ξ_o, a quantity which describes the domain wall thickness. At low temperatures (well below the mean-field T_c) $\xi_o \sim O(1)$. In contrast, the true **correlation length ξ grows exponentially**, but remains finite at low (but finite) temperatures. This behavior is our first example with a more general phenomenon: finite temperature phase transitions are suppressed at or below a certain lower critical dimension d_{LC}, where the fluctuation effects are strong enough. In the case of the classical Ising model $d_{LC} = 1$.

We have already seen that, according to the Ginzburg criterion, mean-field theory becomes exact for $d > d_{UC}$. In lower dimensions the fluctuations become important until they finally completely eliminate any finite temperature ordering for $d \leq d_{LC}$. Thus, nontrivial critical behavior is found only in the range $d_{LC} \leq d < d_{UC}$. Our main task is to understand in detail what happens in this regime.