$2+\varepsilon$ Expansion, Localization, and Asymptotic Freedom

We have seen that in models with continuous symmetry, phase (transverse) fluctuations completely eliminate any finite temperature ordering in dimension $d \leq 2$. This opens a possibility to formulate another perturbative RG approach, leading to the $2+\varepsilon$ expansion method of Polyakov. The structure of the resulting theory bears relevance to several classes of physical systems close to their lower critical expansion.

Nonlinear σ-model revisited

We already argued that the leading low energy excitations in models with continuous symmetry have a character of long-wavelength spin waves. Our estimates showed that the transition temperature decreases as the dimension is reduced towards $d = 2$, such that

$$T_c \sim (d-2).$$

Close to two dimensions, the transition takes place at infinitesimally low temperatures, where the spin wave excitations completely dominate the physics. This immediately suggests a new perturbative approach to constructing the RG theory, based on a $d = 2 + \varepsilon$ expansion. In this regime we expect to have very few spin wave excitations, and their scattering can thus be treated perturbatively. A systematic procedure how this program can be implemented was first developed by Polyakov, who concentrated on the non-linear σ-model given by

$$Z[j] = \int D\boldsymbol{\sigma}(\mathbf{x})\ \delta[\boldsymbol{\sigma}^2(\mathbf{x}) - \mathbf{1}]\ \exp\left\{-\frac{1}{2g}\int d\mathbf{x}\ (\nabla\,\boldsymbol{\sigma}(\mathbf{x}))^2 + \int d\mathbf{x}\ \mathbf{j}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})\right\}.$$

As we recall, the coupling constant $g \sim T/J$, so the theory describing low temperature behavior (where the transition is expected to be found) should be developed by doing perturbation theory in g. Although the Action is quadratic, this calculation is nontrivial due to the constraint $\boldsymbol{\sigma}^2(\mathbf{x}) = \mathbf{1}$,which must be eliminated before setting up a diagrammatic formulation. We consider a general $O(N)$ σ-model where we can use the parametrization $\boldsymbol{\sigma} = (\sigma, \pi_1, ..., \pi_{N-1})$. The longitudinal component σ can be eliminated as

$$\sigma = \sqrt{1-\boldsymbol{\pi}^2} = 1 - \frac{1}{2}\boldsymbol{\pi}^2 - \frac{1}{8}\left(\boldsymbol{\pi}^2\right)^2 + \cdots .$$

We used the vector notation for the transverse components $\boldsymbol{\sigma}= (\sigma, \boldsymbol{\pi})$, such that $\boldsymbol{\sigma}^2 = \sigma^2 + \boldsymbol{\pi}^2$, etc.

Note that the elimination of the constraint produces a Jacobian factor from the integration measure, as follows.

$$\begin{aligned}\int \prod_{\mathbf{x}} D\boldsymbol{\sigma}(\mathbf{x})\ \delta[\boldsymbol{\sigma}^2(\mathbf{x}) - \mathbf{1}] &= \int \prod_{\mathbf{x}} D\boldsymbol{\pi}(\mathbf{x}) \left|\frac{\partial \sigma}{\partial \boldsymbol{\pi}}\right| D\boldsymbol{\pi}(\mathbf{x}) \\ &= \int \prod_{\mathbf{x}} \frac{D\boldsymbol{\pi}(\mathbf{x})}{\sqrt{1-\boldsymbol{\pi}^2}} \\ &= \int D\boldsymbol{\pi}(\mathbf{x}) \exp\{-\frac{1}{2}\rho \int d\mathbf{x}\ \ln\left(1-\boldsymbol{\pi}^2\right)^{1/2}\},\end{aligned}$$

where $\rho = a^d$ is the unit cell volume. The Jacobian factor $\frac{1}{2}\rho \int d\mathbf{x} \ln\left(1-\boldsymbol{\pi}^2\right)^{1/2}$ is added to the Action

$$S = \frac{1}{2g} \int d\mathbf{x} \left\{ \ (\nabla \boldsymbol{\pi})^2 + \left(\nabla \sqrt{1-\boldsymbol{\pi}^2}\right)^2 \right\} - j \int d\mathbf{x} \sqrt{1-\boldsymbol{\pi}^2} + \frac{1}{2}\rho \int d\mathbf{x} \ln\left(1-\boldsymbol{\pi}^2\right)^{1/2}.$$

As we can see, after the constraint has been eliminated, the Action assumes a form nonlinear in the independent $\boldsymbol{\pi}-$fields, hence the name "nonlinear σ-model".

In the weak coupling limit $g \ll 1$, the amplitude of the transverse $\boldsymbol{\pi}$-fluctuations are small, and the leading order contributions can be obtained by expanding the Action in powers of $\boldsymbol{\pi}$. We find

$$S = S_o + S_{int},$$

where the quadratic part, describing non-interacting spin waves

$$S_o = \frac{1}{2g} \int d\mathbf{x}\ \left\{(\nabla \boldsymbol{\pi})^2 + gj\boldsymbol{\pi}^2\right\},$$

and the nonlinear terms, describing their interactions

$$S_{int} = \frac{1}{2g} \int d\mathbf{x}\ \left\{(\boldsymbol{\pi}\cdot\nabla \boldsymbol{\pi})^2 + gj\,(\boldsymbol{\pi} \cdot \boldsymbol{\pi})^2 - g\rho\boldsymbol{\pi}^2\right\}.$$

[The contribution $-g\rho\boldsymbol{\pi}^2$ form the Jacobian factor can be treated as a perturbation, as it only cancels out all the terms in perturbation theory that would generate the mass under renormalization.]

Written in this way, the structure of the theory is very similar to the already familiar ϕ^4 theory, and we can again use the Wicks theorem and Feynmann diagrams to set up a perturbaton theory calculation. For example, the free propagator is

$$G_o(q) = \frac{g}{k^2 + h}.$$

Note that, in contrast to the ordinary Gaussian model, the external field j now plays a role of the mass term. Physically, this reflects the fact that the field suppresses the fluctuations transverse to its direction, and which corresponds to spin waves.

In this perturbation theory, each propagator line carries one power of the (small) coupling constant g, while each spin-wave interaction vertex $\frac{1}{g}\left(\boldsymbol{\pi}\cdot\nabla\,\boldsymbol{\pi}\right)^2$ is inversely proportional to g.

2+ϵ Expansion

We are now in a position to apply the momentum-shell RG approach precisely as in the ϕ^4 theory. As in illustration, we only discuss the renormalization of the coupling constant at $j = 0$. As before, we perform the cumulant expansion to first order in S_{int}, and find (**problem 5.5**; see also the book by Chaikin and Lubensky, or the lecture notes by Ben Simmons)

$$\beta_g = \frac{dg}{d\ell} = -\varepsilon g + \frac{(N-2)}{2\pi}\Lambda^{d-2} g^2 + O(g^3).$$

The critical fixed point is, as usual found from the zero of the beta-function

$$g^* = \frac{2\pi\Lambda^{d-2}\varepsilon}{N-2}.$$

As we expected, the critical temperature

$$T_c \sim g^* \sim \frac{\varepsilon}{N-2} = \frac{d-2}{N-2}.$$

Linearizing the beta-function around the fixed point

$$\frac{dg}{d\ell} \approx \varepsilon\left(g - g^*\right),$$

and the correlation length exponent

$$\nu = \frac{1}{\varepsilon}.$$

Similar results can be obtained for the field renormalization and from this all the other exponents can be calculated (**problem 5.5).** As it turns out, the results obtained by exptrapolating these to $d = 3$ (i.e. by setting $\varepsilon = 1$) are not accurate at all, in contrast to those obtained from a $4 - \varepsilon$ expansion. The situation does not improve at all if higher order corrections in ε are added. In general, there seems to be very poor convergence of the $2 + \varepsilon$ expansion, when extrapolated to $d = 3$.

We should emphasize that these results are perfectly rigorous and exact, provided that $\varepsilon = d - 2 \ll 1$. It is interesting that the critical exponent ν is independent of the number of components N of the order parameter. In contrast, note that the critical temperature increases when N is reduced. This effect is especially important for the XY model ($N = 2$), where the result seems to suggest that $T_c = \infty$! This puzzling result deserves comment. When we look closer, we can see that the perturbative corrections that come from the non-linear terms all have the prefactor. $(N - 2)$. Physically, the spin-wave scattering processes, that are effective in reducing T_c for $N > 2$, are not operative in the XY limit. Here, careful calculations show that order by order, all corrections vanish order by order in perturbation theory. The reason for this is that for the XY model, excitations other then spin waves prove responsible for leading thermal fluctuations, which have a character of vortices (in $d = 2$) of vortex loops ($d > 2$). We will return to these interesting issues when we discuss the Kosterlitz-Thouless transition of the $d = 2$ XY model.

Behavior at the lower critical dimension

It is interesting to analyze our RG results precisely at the lower critical dimension $d_{lc} = 2$, concentrating on $N > 2$ models, where spin-waves dominate the physics. Here

$$\beta_g = \frac{dg}{d\ell} = \frac{(N-2)}{2\pi} g^2 + O(g^3).$$

As we can see, the unstable fixed point now is precisely at weak coupling $g^* = 0$. Physically, this corresponds to the critical point located precisely at $T = 0$. For an arbitrarily small bare value of g (i.e. small physical temperature), $g(\ell)$ grows without bounds under renormalization, and we "flow" to the $T = \infty$ (paramagnetic) fixed point. We conclude that there is no order at any finite temperature.

It is also interesting to determine the temperature dependence of the correlation length. We first solve the differential equation

$$\frac{dg}{d\ell} = Ag^2,$$

where $A = (N-2)/2\pi$. We find

$$\frac{1}{g(\ell)} = \frac{1}{g_o} - A\ell = \frac{1}{g_o} - A\ln b,$$

with the length-rescaling factor $b = e^{\ell}$, or

$$g(b) = \frac{g_o}{1 - g_o A \ln b}.$$

Since $g(b) \to \infty$,

$$b \approx \frac{2\pi}{N-2} \exp\{1/g_o\}.$$

Next, we use the Kadanoff scaling relation

$$\xi(g_o) = b\xi(g(\ell)),$$

and the fact that $\xi(T = \infty) \sim 1$, and $g_o \sim T/J$

$$\xi(T) \sim \frac{1}{N-2} \exp\{aJ/T\},$$

where a is a constant of order one.

Similarly as in the $d = 1$ Ising model (which was also at its lower critical dimension and an exact RG treatment was available), we find that the correlation length diverges exponentially as the temperature is reduced. Similar results are found in other problems near the lower critical dimension. One example is the problem of Anderson localization of electrons in a random potential. here, the electrons form a bound state with impurities for strong enough disorder, and become localized, with localization radius ξ. For weak enough disorder, in contrast, the electrons are not bound, and travel through the system. As in the $O(N)$ magnet, this system also has its lower critical dimension $d_{lc} = 2$. In dimension $d \leq 2$ the electrons are localized no matter how weak the disorder is, although the localization radius can be very large. Precisely in two dimensions, the localization radius is again exponentially large. This behavior has been explained using the scaling theory of localization (Abrahams, Anderson, Licciardello, and Ramakrishnan, 1979), and the non-linear σ-model approach of Wegner (1980). This approach is particularly attractive for the problem of the metal-insulator transition, where an order parameter and an appropriate spontaneous symmetry breaking scheme is not available. To describe the critical behavior, the $2 + \varepsilon$ approach is thus more convenient, as opposed to the $4 - \varepsilon$ approach which is an expansion around a Landau (mean-field) solution.

Asymptotic freedom

Another problem that has formal similarities to the behavior of the $O(N)$ magnet is that of quark confinement and asymptotic freedom within quantum chromodynamics (QCD). In this theory, quarks interact through the exchange of bosonic particles called "gluons". This theory can be considered as an extension of ordinary quantum electrodynamics (QED), where photons mediate the Coulomb force between electrons and positrons. As a result of the non-Abelian structure of QCD, and in contrast to photons in QED, the gluons are not non-interacting particles even in absence of quarks. Therefore, this "pure gauge" version of the theory is an interacting theory, that is nontrivial to solve. The solution of this simplified problem is nevertheless important, as the form of the gluon propagator determines the effective quark-quark interaction potential. Detailed calculations have determined that the phenomenon of quark confinement reflect the dramatic increase of the confining potential as the distance is increased. In contrast, the quark-quark interaction proves to be very weak when they are close to each other. The discovery of this fascinating mechanism by Gross and Wilczek, and independently by Politzer (both papers in the same 1973 issue of Phys. Rev. Lett.) lead to the 2005 Nobel Prize in physics.

Technically, this ground breaking calculation determined the (length or momentum) scale dependence of the gluon coupling constant g. In absence of quarks, the Lagrangian describing gluons takes the form

$$\mathcal{L} = -\frac{1}{4} F^{\alpha}_{\mu\nu}(x) F^{\mu\nu}_{\alpha}(x),$$

where

$$F^{\alpha}_{\mu\nu}(x) = \partial_\mu A^{\alpha}_{\nu}(x) - \partial_\nu A^{\alpha}_{\mu}(x) + g C^{\alpha}_{\alpha\beta\gamma} A^{\beta}_{\mu}(x) A^{\gamma}_{\nu}(x),$$

$C^{\alpha}_{\alpha\beta\gamma}$ are the $SU(3)$ structure constants, and $A^{\alpha}_{\nu}(x)$ are the gauge fields describing the gluons, which are the analogue of the vector potential in QED. As we can see, for $g = 0$, the Lagrangian is quadratic in the gauge fields, representing freely propagating gluons. For $g \neq 0$, the gluons interact and one has a many-body problem at hand. Assuming that the coupling constant g is small, perturbative (Feynmann diagram) calculations can be performed, similar as in QED. But is it small? Well...it depends on the energy (momentum) scale one considers. As in the statistical physics, the proper technical approach to determine the scale dependence of the coupling constant is through the perturbative renormalization group calculations.

Similarly as in the $d = 2$ $O(N)$ model, they find for $d = 4$ QCD the only fixed point to exist at $g^* = 0$, were the beta function has non-linear g-dependence. As a result, they find the beta function of the form

$$\beta_g \sim g^3,$$

giving the momentum dependent coupling constant

$$g^2(k) = \frac{g^2}{1 + g^2 B \ln(k^2/M^2)},$$

where B is a constant. This results indicates that $g(k) \to 0$ when $k \to \infty$, i.e. that at large momenta the effective coupling constant becomes very small. Since 1973 this ground-breaking result was confirmed by numerous experiments.