Homework Assignment #10

Critical and Crossover Behavior of the $O(n)$ Model

The following problems contain more challenging questions which are indicated by (*). I expects students in Condensed Matter Theory to complete these problems, since they need to become fluent in doing such technical calculations.

Problem 1. Consider a perturbation breaking the $O(n)$ symmetry (describing the single ion anisotropy), of the form

$$\delta S_g = \frac{1}{2} g \int d\mathbf{x}\ \phi_n^2(\mathbf{x}).$$

Here, $\phi_n(\mathbf{x})$ is the n-th component of the vector field $\phi_\alpha(\mathbf{x})$ ($\alpha = 1, ..., n$) . By separating the components $\alpha = 1, ..., n-1$ and $\alpha = n$, action then can be written as

$$S[\phi] = \frac{1}{2}\sum_{\alpha=1}^{n-1} \int d\mathbf{x}\, \phi_\alpha(\mathbf{x})\, [r - \nabla^2]\, \phi_\alpha(\mathbf{x}) + \frac{1}{2} \int d\mathbf{x}\, \phi_n(\mathbf{x})\, [r + g - \nabla^2]\, \phi_n(\mathbf{x}) +$$
$$+ \frac{u}{4} \int d\mathbf{x} \sum_{\alpha,\beta=1}^{n-1} \phi_\alpha^2(\mathbf{x}) \phi_\alpha^2(\mathbf{x}) + \frac{v}{4} \int d\mathbf{x} \sum_{\alpha=1}^{n-1} \phi_\alpha^2(\mathbf{x}) \phi_n^2(\mathbf{x}) + \frac{w}{4} \int d\mathbf{x}\ \phi_n^2(\mathbf{x}) \phi_n^2(\mathbf{x}).$$

Note that the propagator of the $\alpha = n$ components now has "mass" $r+g$, while all the other components still have "mass" r. Diagrammatically, we need to consider diagrams with two different types of propagators. Note that the "bare" values of the interaction amplitudes u, v, and w are all the same and equal to u. However, the three terms differ by symmetry, since in presence of the g-perturbation the channel $\alpha = n$ is different from the other channels. Therefore, we expect the three terms to renormalize differently under RG, and we thus consider three different types of vertices as well. To simplify the notation, also rescale the interaction amplitude to absorb the factors Ω_d that arise from momentum shell integration.

The RG equations in presence of this perturbation take the form

$$\begin{aligned}
\frac{dr}{d\ell} &= 2r + (n+1)\frac{u}{r+\Lambda^2} + \frac{v}{r+g+\Lambda^2};\\
\frac{d(r+g)}{d\ell} &= 2(r+g) + (n-1)\frac{u}{r+\Lambda^2} + 3\frac{v}{r+g+\Lambda^2};\\
\frac{du}{d\ell} &= \varepsilon u - (n+7)\frac{u^2}{(r+\Lambda^2)^2} - \frac{v^2}{(r+\Lambda^2)^2};\\
\frac{dv}{d\ell} &= \varepsilon v - (n+1)\frac{uv}{(r+\Lambda^2)^2} - 3\frac{vw}{(r+g+\Lambda^2)^2} - 4\frac{v^2}{(r+g+\Lambda^2)(r+\Lambda^2)};\\
\frac{dw}{d\ell} &= \varepsilon v - (n-1)\frac{v^2}{(r+\Lambda^2)^2} - 9\frac{w^2}{(r+g+\Lambda^2)^2}.
\end{aligned}$$

Find the new fixed points in presence of this perturbation. To do this, note that g grows under RG whenever the bare value $g \neq 0$.

(a) Consider first $g > 0$. In some terms of the RG equations g shows up in the denominator, and we can drop all such terms when $g \longrightarrow \infty$. Show that the resulting RG equations are identical to those of an $n-1$ component isotropic magnet (this is a so-called "XY model" for the Heisenberg $n = 3$ case).

(b) For the initial value $g < 0$, under iteration $g \longrightarrow -\infty$. In this limit we can find a fixed point that satisfies $r \longrightarrow +\infty$, but with a condition that the combination $(r+g) \sim O(\varepsilon)$ remains finite, and $u = v = 0$. In this limit we can drop all the terms that have only r in the denominator. Show that in this limit the RG equations reduce to those for an Ising magnet.

Problem 2*. Derive the RG equations for the $O(n)$ nonlinear σ-model, using the $2+\varepsilon$ expansion and the momentum shell RG technique. Calculate the values for the exponents ν, β, an γ for the Heisenberg model ($n = 3$) by setting $\varepsilon = 1$. Compare these results to those obtained for the same model using the $4-\varepsilon$ expansion, and the exact results (see literature). Which ε-expansion approach works better?

Problem 3*. Carry out the Hubbard-Stratonovich transformation outlined in the lecture to solve the $O(n)$ model in the $n \longrightarrow \infty$ limit. Using this approach calculate the order parameter exponent γ. Show that for $d > 4$ it reduces to the Landau (mean-field) prediction $\gamma = 1$, but that for $2 < d < 4$ one obtains different critical behavior. How does the resulting value of γ for $d = 3$ compare to the results obtained using the $4-\varepsilon$ and the $2+\varepsilon$ expansion approaches?