Phase Transitions - Homework 10

Problem 10.1

Linearizing the obtained RG equations in small g at the isotropic Wilson-Fisher fixed point $u = v = w = u_{\text{iso}}$ gives:

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}l} &= 2r + (n+2)\frac{u_{\text{iso}}}{r+\Lambda^2} - g\frac{u_{\text{iso}}}{(r+\Lambda^2)^2} \\
\frac{\mathrm{d}(r+g)}{\mathrm{d}l} &= 2(r+g) + (n-1)\frac{u_{\text{iso}}}{r+\Lambda^2} - 3g\frac{u_{\text{iso}}}{(r+\Lambda^2)^2} \\
\frac{\mathrm{d}u}{\mathrm{d}l} &= \varepsilon u_{\text{iso}} - (n+8)\frac{u_{\text{iso}}^2}{(r+\Lambda^2)^2} \\
\frac{\mathrm{d}v}{\mathrm{d}l} &= \varepsilon u_{\text{iso}} - (n+8)\frac{u_{\text{iso}}^2}{(r+\Lambda^2)^2} + 14g\frac{u_{\text{iso}}^2}{(r+\Lambda^2)^3} \\
\frac{\mathrm{d}w}{\mathrm{d}l} &= \varepsilon u_{\text{iso}} - (n+8)\frac{u_{\text{iso}}^2}{(r+\Lambda^2)^2} + 18g\frac{u_{\text{iso}}^2}{(r+\Lambda^2)^3}
\end{aligned}$$

From the first two lines we obtain $\frac{\mathrm{d}r}{\mathrm{d}l} = 2g(1 - \frac{u_{\text{iso}}}{r+\Lambda^2})$, so if $g = 0$ it stays at zero, but for $g \neq 0$ it keeps growing exponentially. The last three lines clearly tell us that u, v, and w behave different under renormalization if $g \neq 0$, so isotropy will be destroyed.

(a)

For $g > 0$ the RG running drives g up fast. Looking at the RG equations in the limit $g \to \infty$ we find:

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}l} &= 2r + (n+1)\frac{u}{r+\Lambda^2} \\
\frac{\mathrm{d}u}{\mathrm{d}l} &= \varepsilon u - (n+7)\frac{u^2}{(r+\Lambda^2)^2} - \frac{v^2}{(r+\Lambda^2)^2} \\
\frac{\mathrm{d}v}{\mathrm{d}l} &= \varepsilon v - (n+1)\frac{uv}{(r+\Lambda^2)^2} \\
\frac{\mathrm{d}w}{\mathrm{d}l} &= \varepsilon w - (n-1)\frac{v^2}{(r+\Lambda^2)^2}
\end{aligned}$$

For $v = w = 0$ this reduces to

$$\frac{\mathrm{d}r}{\mathrm{d}l} = 2r + (n+1)\frac{u}{r+\Lambda^2}$$
$$\frac{\mathrm{d}u}{\mathrm{d}l} = \varepsilon u - (n+7)\frac{u^2}{(r+\Lambda^2)^2}$$

Which is nothing else than the result of Problem 9.1 with $n-1$ instead of n, i.e. the $n-1$ component isotropic magnet is recovered. Accordingly the fixed point is at

$$u^\star_{\mathrm{WF}} = \frac{3\varepsilon}{(n+7)b} \qquad \text{and} \qquad r^\star_{\mathrm{WF}} = -\frac{n+1}{2(n+7)}\frac{a}{b}\varepsilon \qquad (g \to \infty, v = w = 0).$$

(b)

For $g < 0$ the RG running drives g to large negative numbers. There is a fixed point at $r \to \infty$, such that $\tilde{r} \equiv (r+g) \sim \mathcal{O}(\varepsilon)$. In this limit the RG equations are

$$\frac{\mathrm{d}(\tilde{r})}{\mathrm{d}l} = 2\tilde{r} + 3\frac{w}{\tilde{r}+\Lambda^2}$$
$$\frac{\mathrm{d}u}{\mathrm{d}l} = \varepsilon u - \frac{v^2}{(r+\Lambda^2)^2}$$
$$\frac{\mathrm{d}v}{\mathrm{d}l} = \varepsilon v - 3\frac{vw}{(\tilde{r}+\Lambda^2)^2}$$
$$\frac{\mathrm{d}w}{\mathrm{d}l} = \varepsilon w - 9\frac{w^2}{(\tilde{r}+\Lambda^2)^2}$$

For $u = v = 0$ this reduces to

$$\frac{\mathrm{d}\tilde{r}}{\mathrm{d}l} = 2\tilde{r} + 3\frac{w}{\tilde{r}+\Lambda^2}$$
$$\frac{\mathrm{d}w}{\mathrm{d}l} = \varepsilon w - 9\frac{w^2}{(\tilde{r}+\Lambda^2)^2}$$

This the RG equations of the Ising magnet derived in class. The fixed point is at

$$w^\star_{\mathrm{WF}} = \frac{\varepsilon}{3b} \qquad \text{and} \qquad \tilde{r}^\star_{\mathrm{WF}} = -\frac{a}{6b}\varepsilon \qquad (g \to -\infty, r \to \infty, u = v = 0).$$