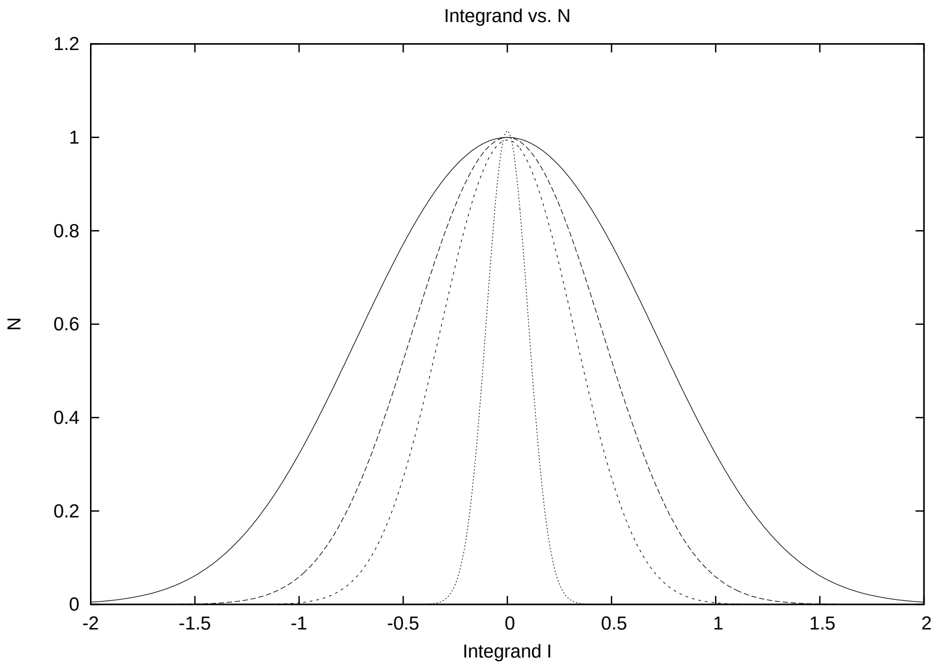
Problem 1.

The partition function for an infinite range Ising ferromagnet can be written

$$Z(T,h) = \left(\frac{\beta J}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dt \exp\left\{N[-\frac{t^2}{2\beta J} + \ln(\cosh(\beta h + t)) + \ln 2]\right\}. \quad (1)$$

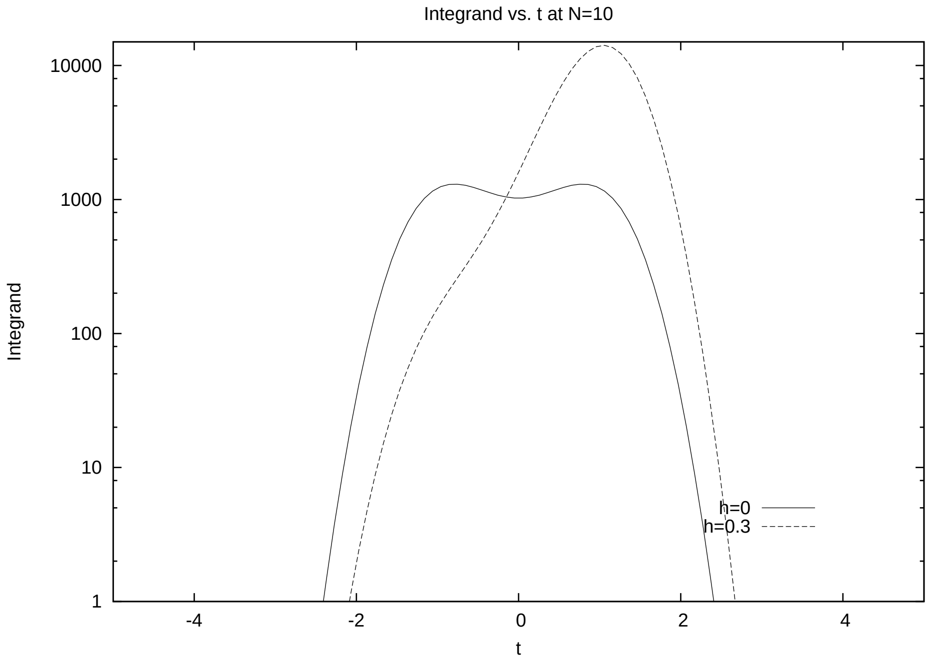
Now, to begin computing the integral using the saddle point method (also called the method of steepest descent) let's plot the integrand.

a.)You can see below that for larger and larger N at $h = 0$ and $\beta J = 0.5$, the integrand grows sharper around zero, and would become a delta spike as N.grows larger. This is why in the $N \to \infty$ limit the saddle point method is exact.



b.) Then for $h = 0$ and $\beta J = 1.2$ at $N = 10$ (in the plot above as well) the integrand developes two degenerate peaks. The peaks emerge for any $J > T$.

c.) With the external field on, one can see that the degeneracy of the peaks is destroyed, and the integrand again developes only one sharp peak. The effect of the field is to break the degeneracy.



d.) The argument of the exponetial is $f\,(t,h) = -\frac{t^2}{2\beta J} + \ln(\cosh(\beta h + t)) + \ln 2$, and the maxima occur at

$$\frac{df}{dt} = 0 = -\frac{t}{\beta J} + \tanh\left(\beta h + t\right), \tag{2}$$

which has solutions $t = t_{\pm}$, if one solves the trancendental equation above. Also,

$$\frac{d^2 f}{dt^2} = -\frac{1}{\beta J} + \text{sech}^2\left(\beta h + t\right). \tag{3}$$

Now expanding f to second order around each maximia gives

$$f_{+}\left(t,h\right) = -\frac{t_{+}}{\beta J} + \tanh\left(\beta h + t_{+}\right) + \frac{1}{2}\left(-\frac{1}{\beta J} + \text{sech}^2\left(\beta h + t_{+}\right)\right)\left(t - t_{+}\right)^2 \tag{4}$$

$$f_{\text{-}}\left(t,h\right) = -\frac{t_{-}}{\beta J} + \tanh\left(\beta h + t_{-}\right) + \frac{1}{2}\left(-\frac{1}{\beta J} + \text{sech}^2\left(\beta h + t_{-}\right)\right)\left(t - t_{-}\right)^2 \tag{5}$$

Then the integral for the positive and negative saddle points gives

$$I_{\pm} = \int_{-\infty}^{\infty} dt \exp\left(f_{\pm}\left(t,h\right)\right) \tag{6}$$

$$= \exp\left(-\frac{t_{\pm}N}{\beta J} + N\tanh\left(\beta h + t_{\pm}\right)\right)\sqrt{\frac{2\pi}{N\left(-\frac{1}{\beta J} + \text{sech}^2\left(\beta h + t_{\pm}\right)\right)}}, \tag{7}$$

so that the full solution is

$$I = \exp\left(-\frac{t_{+}N}{\beta J} + N\tanh\left(\beta h + t_{+}\right)\right)\sqrt{\frac{2\pi}{N\left(-\frac{1}{\beta J} + \text{sech}^2\left(\beta h + t_{+}\right)\right)}} \tag{8}$$

$$+ \exp\left(-\frac{t_{-}N}{\beta J} + N\tanh\left(\beta h + t_{-}\right)\right)\sqrt{\frac{2\pi}{N\left(-\frac{1}{\beta J} + \text{sech}^2\left(\beta h + t_{-}\right)\right)}}. \tag{9}$$

In the limit that $N \to \infty$, the peaks become very sharp and the integral is exact.

e.)

In general for an integral $I(N) = \int dt \exp\left(-Nf\left(t\right)\right)$, with mimima at $t = t_0$ we can write

$$I(N) = e^{-Nf(t_0)} \int dt \exp\left(-\frac{N}{2} f_0''(t-t_0)^2 - N \sum_{n=3}^{\infty} \frac{f_0^{(n)}}{n!}(t-t_0)^n \right). \tag{10}$$

Making a change of variables, let $\sqrt{N f_0''}(t - t_0) = z$, so that the integral can be written as

$$I(N) = e^{-Nf(t_0)} \sqrt{\frac{2\pi}{N f_0''}} \int \frac{dz}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} z^2 - \sum_{n=3}^{\infty} \frac{f_0^{(n)}}{n!\,N^{\frac{n}{2}-1}} \frac{z^n}{(f_0'')^{\frac{n}{2}}} \right), \tag{11}$$

that is the first order term is normalized to unity. First write the exponetial as the product of the exponetials of the quadratic term and the infinite sum. Then for N large we can expand that exponetial in powers of N^{-1} which is small.

$$S = \exp\left(-\sum_{n=3}^{\infty} \frac{f_0^{(n)}}{n!\,N^{\frac{n}{2}-1}} \frac{z^n}{(f_0'')^{\frac{n}{2}}} \right) \tag{12}$$

$$= 1 - \sum_{n=3}^{\infty} \frac{f_0^{(n)}}{n!\,N^{\frac{n}{2}-1}} \frac{z^n}{(f_0'')^{\frac{n}{2}}} + \frac{1}{2} \left(\sum_{n=3}^{\infty} \frac{f_0^{(n)}}{n!\,N^{\frac{n}{2}-1}} \frac{z^n}{(f_0'')^{\frac{n}{2}}} \right) \left(\sum_{m=3}^{\infty} \frac{f_0^{(m)}}{m!\,N^{\frac{m}{2}-1}} \frac{z^m}{(f_0'')^{\frac{m}{2}}} \right) + O\left(\Sigma^3\right) \tag{13}$$

$$= 1 - \frac{f_0^{(3)}}{3!\,N^{\frac{1}{2}}} \frac{z^3}{(f_0'')^{\frac{3}{2}}} - \frac{f_0^{(4)}}{4!\,N} \frac{z^4}{(f_0'')^2} - \frac{f_0^{(5)}}{5!\,N^{\frac{3}{2}}} \frac{z^5}{(f_0'')^{\frac{5}{2}}} - \frac{f_0^{(6)}}{6!\,N^2} \frac{z^6}{(f_0'')^3} \tag{14}$$

$$+ \frac{1}{2} \left(\frac{f_0^{(3)}}{3!\,N^{\frac{1}{2}}} \frac{z^3}{(f_0'')^{\frac{3}{2}}} + \frac{f_0^{(4)}}{4!\,N} \frac{z^4}{(f_0'')^2} + \frac{f_0^{(5)}}{5!\,N^{\frac{3}{2}}} \frac{z^5}{(f_0'')^{\frac{5}{2}}} \right)^2 \tag{15}$$

$$= 1 - \frac{f_0^{(3)}}{3!\,N^{\frac{1}{2}}} \frac{z^3}{(f_0'')^{\frac{3}{2}}} - \frac{f_0^{(4)}}{4!\,N} \frac{z^4}{(f_0'')^2} - \frac{f_0^{(5)}}{5!\,N^{\frac{3}{2}}} \frac{z^5}{(f_0'')^{\frac{5}{2}}} + \left(\frac{1}{2} \frac{\left(f_0^{(3)}\right)^2}{3!3!N} \frac{z^6}{(f_0'')^3} \right) + O\left(z^7\right). \tag{16}$$

Now integrating the gaussian integrals using the expanded S, we find

$$I(N) = e^{-Nf(t_0)} \sqrt{\frac{2\pi}{N f_0''}} \int \frac{dz}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} [1 - \frac{f_0^{(3)}}{3!\,N^{\frac{1}{2}}} \frac{z^3}{(f_0'')^{\frac{3}{2}}} - \frac{f_0^{(4)}}{4!\,N} \frac{z^4}{(f_0'')^2} \tag{17}$$

$$- \frac{f_0^{(5)}}{5!\,N^{\frac{3}{2}}} \frac{z^5}{(f_0'')^{\frac{5}{2}}} + \left(\frac{1}{2} \frac{\left(f_0^{(3)}\right)^2}{3!3!N} \frac{z^6}{(f_0'')^3} \right)] \tag{18}$$

$$= e^{-Nf(t_0)} \sqrt{\frac{2\pi}{N f_0''}} [1 - \frac{3 f_0^{(4)}}{4!\,N} \frac{1}{(f_0'')^2} + \frac{15}{2} \frac{\left(f_0^{(3)}\right)^2}{3!3!N} \frac{1}{(f_0'')^3} + O\left(\frac{1}{N^2}\right)] \tag{19}$$

$$= e^{-Nf(t_0)} \sqrt{\frac{2\pi}{N f_0''}} \exp\left(\frac{1}{N} (\frac{5}{24} \frac{\left(f_0^{(3)}\right)^2}{(f_0'')^3} - \frac{f_0^{(4)}}{8\,(f_0'')^2}) \right) \tag{20}$$

$$= \exp\left(-Nf_0 + \frac{1}{2} \ln\left(\frac{2\pi}{N f_0''} \right) + \frac{1}{N} (\frac{5}{24} \frac{\left(f_0^{(3)}\right)^2}{(f_0'')^3} - \frac{f_0^{(4)}}{8\,(f_0'')^2}) \right). \tag{21}$$

With this general expression we can simply take a few derivatives and "plug-in" the results for our integral. The result is

$$I = \exp\left(-N\left(\tanh t_+ - \frac{t_+}{\beta J}\right) + \frac{1}{2}\ln\left(\frac{2\pi}{N\left(\operatorname{sech}^2(t_+) - \frac{1}{\beta J}\right)}\right)\right) \tag{22}$$

$$* \exp\left(\frac{1}{N}\left(\frac{5}{24}\frac{\left(4\operatorname{sech}^4(t_+)\right)\tanh^2(t_+)}{\left(\operatorname{sech}^2(t_+) - \frac{1}{\beta J}\right)^3} - \frac{4\operatorname{sech}^2(t_+)\tanh^2(t_+) - 2\operatorname{sech}^4(t_+)}{8\left(\operatorname{sech}^2(t_+) - \frac{1}{\beta J}\right)^2}\right)\right) \tag{23}$$

$$+ \exp\left(-N\left(\tanh t_- - \frac{t_-}{\beta J}\right) + \frac{1}{2}\ln\left(\frac{2\pi}{N\left(\operatorname{sech}^2(t_-) - \frac{1}{\beta J}\right)}\right)\right) \tag{24}$$

$$* \exp\left(\frac{1}{N}\left(\frac{5}{24}\frac{\left(4\operatorname{sech}^4(t_-)\right)\tanh^2(t_-)}{\left(\operatorname{sech}^2(t_-) - \frac{1}{\beta J}\right)^3} - \frac{4\operatorname{sech}^2(t_-)\tanh^2(t_-) - 2\operatorname{sech}^4(t_-)}{8\left(\operatorname{sech}^2(t_-) - \frac{1}{\beta J}\right)^2}\right)\right). \tag{25}$$

Problem 2.
a.)

The Hamiltonian for an antiferromagnetic Ising magnet on a bipartite lattice is given by

$$H = \frac{J}{2}\sum_{<ij>} S_i S_j - h\sum_i S_i \tag{26}$$

Now we make the mean field approximation, that is each spin on lattice A feels an average magnetization due to spins on lattice B, and vice versa. Therefore the magnetization of each sublattice is

$$m_A = \left\langle S_i^A \right\rangle ; m_B = \left\langle S_i^B \right\rangle \tag{27}$$

To decouple the spins, use the Weiss mean-field approximation to find, the energy of each spin $i \in A$.

$$E_i^A = \frac{J}{2} S_i \sum_{j\in B}^{z} \left\langle S_j^B \right\rangle - hS_i \tag{28}$$

$$= \left(\frac{J}{2} z m_B - h\right) S_i = B_{eff}^B S_i \tag{29}$$

and similarly for $E_j^B = \frac{J}{2} S_j z m_A - hS_j = B_{eff}^A\ S_j$. These relations give us the conditions of the magnetization

$$m_A = \left\langle S_i^A \right\rangle = Z^{-1}\sum_{\{S\}} S_i e^{-\beta\left(\frac{J}{2}zm_B - h\right)S_i} \tag{30}$$

$$= \tanh\left(-\frac{J\beta}{2} z m_B + h\beta\right) \tag{31}$$

and similaryly for, $m_B = \tanh\left(-\frac{J\beta}{2} z m_A + h\beta\right)$. We can now express the average magnetization $m \equiv \frac{1}{2}(m_A + m_B)$, and the staggered magnetization $m^\dagger \equiv \frac{1}{2}(m_A - m_B)$ in the following form.

$$m = \frac{1}{2}\left(\tanh\left(-\frac{J\beta}{2} z\left(m - m^\dagger\right) + h\beta\right) + \tanh\left(-\frac{J\beta}{2} z\left(m + m^\dagger\right) + h\beta\right)\right) \tag{32}$$

$$m^\dagger = \frac{1}{2}\left(\tanh\left(-\frac{J\beta}{2} z\left(m - m^\dagger\right) + h\beta\right) - \tanh\left(-\frac{J\beta}{2} z\left(m + m^\dagger\right) + h\beta\right)\right) \tag{33}$$

b.)

Lets assume that $h = 0$ so that $m = 0 \to m_A = -m_B$. From the second equation above, we find

$$2m^{\dagger} = \tanh\left(\frac{J\beta}{2}zm^{\dagger}\right) - \tanh\left(-\frac{J\beta}{2}zm^{\dagger}\right) \text{;Expanding in } m^{\dagger} \text{ to third order} \tag{34}$$

$$2m^{\dagger} = 2(\frac{J\beta}{2}zm^{\dagger} - \frac{\left(\frac{J\beta}{2}zm^{\dagger}\right)^3}{3}) \tag{35}$$

$$1 = \frac{J\beta}{2}z - \frac{\left(\frac{J\beta}{2}z\right)^3 m^{\dagger 2}}{3} \to \tag{36}$$

$$m^{\dagger} = \frac{1}{\left(\frac{J\beta}{2}z\right)}\sqrt{3\left(1 - \frac{2T}{Jz}\right)} = \frac{2T}{Jz}\sqrt{3\left(\frac{2Jz}{2Jz} - \frac{2T}{Jz}\right)} \tag{37}$$

$$= \frac{2T}{Jz}\sqrt{\frac{2}{Jz}}\sqrt{3\left(\frac{Jz}{2} - T\right)} = T\left(\frac{2}{Jz}\right)^{\frac{3}{2}}\sqrt{3\left(T_N - T\right)}. \tag{38}$$

We see the exponent $\beta = \frac{1}{2}$, as is to be expected since this is a mean field calculation.

c.)

We would now like to find the uniform spin sucseptibility χ,. therefore let $h > 0$. Here m will be small, expand now the first equation from the set above to first order in m and h.

$$m = \frac{1}{2}\left(\tanh\left(-\frac{J\beta}{2}z\left(m - m^{\dagger}\right) + h\beta\right) + \tanh\left(-\frac{J\beta}{2}z\left(m + m^{\dagger}\right) + h\beta\right)\right) \tag{39}$$

$$2m = \left(\beta - \beta\tanh^2\left(\frac{1}{2}Jz\beta m^{\dagger}\right)\right)h + Jz\beta m\left(-1 + \tanh^2\left(\frac{1}{2}Jz\beta m^{\dagger}\right)\right) \tag{40}$$

Solving for m gives

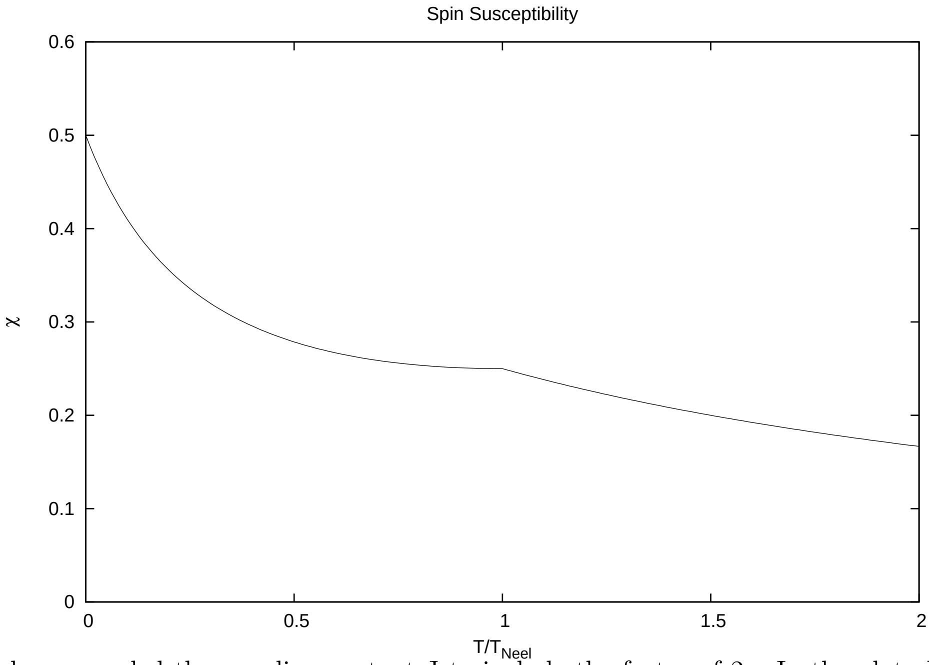
$$m\left(h\right) = \frac{\left(\beta - \beta\tanh^2\left(\frac{1}{2}Jz\beta m^{\dagger}\right)\right)h}{2 + Jz\beta m\left(1 - \tanh^2\left(\frac{1}{2}Jz\beta m^{\dagger}\right)\right)} \tag{41}$$

Therefore the spin sucseptibility χ is given by

$$\chi\left(h\right) = \frac{\left(\beta - \beta\tanh^2\left(\frac{1}{2}Jz\beta m^{\dagger}\right)\right)}{2 + Jz\beta\left(1 - \tanh^2\left(\frac{1}{2}Jz\beta m^{\dagger}\right)\right)}. \tag{42}$$

We can now plug in our expression for the staggered magnetization

$$\chi\left(h\right) = \lim_{h\to 0}\frac{dm\left(h\right)}{dh} = \begin{cases} \frac{\left(1 - \tanh^2\left(\frac{1}{2}Jz\left(\frac{2}{Jz}\right)^{\frac{3}{2}}\sqrt{3|T_N - T|}\right)\right)}{2T + Jz\left(1 - \tanh^2\left(\frac{1}{2}Jz\left(\frac{2}{Jz}\right)^{\frac{3}{2}}\sqrt{3|T_N - T|}\right)\right)}; T < T_N \\ \frac{1}{T + Jz}; T > T_N \end{cases}. \tag{43}$$



I have rescaled the coupling contant J to include the factor of 2. In the plot above one can clearly see the cusp at the Nèel temperature. One could more rigorously show the existance of the cusp, but finding that $\frac{d\chi}{dT}|_{T\rightarrow T_N^-} - \frac{d\chi}{dT}|_{T\rightarrow T_N^+} = const.$