Homework Assignment #2.

In the following, the more challenging questions are indicated by (*). These questions are required work only for the theory students.

Problem 2.1 The partition function for an infinite-range Ising ferromagnet can be written as

$$Z(T,h) = \left(\frac{\beta J}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dt \, \exp\left\{N\left(-\frac{t^2}{2\beta J} + \ln\cosh\{\beta h + t\} + \ln 2\right)\right\}.$$

This integral can for given T and h be calculated numerically, but for N large, we can use the saddle-point method.

(a) To get a feeling how the method works, use a computer to plot the integrand

$$I = \exp\left\{N\left(-\frac{t^2}{2\beta J} + \ln\cosh\{\beta h + t\} + \ln 2\right)\right\},$$

for $N = 2, 4, 6$; $\beta J = 0.5$; and $h = 0$. Where is the integrand peaked? What happens as N is increased?

(b) What happens if we now turn on the external field, and plot for $N = 10$; $\beta J = 1.2$; and $h = 0.3$? Explain the effect of the field.

(c) Now use the approach from lecture notes to obtain an expression for the magnetization (taking the derivative of the free energy with respect to field, etc.), in terms of a similar integral. Use the computer to calculate the magnetization as a function fo field for $\beta J = 1.2$, for $N = 2, 4, 6$. Explain how increasing N at any $h \neq 0$ leads to finite magntization in the $N \to \infty$ limit, while taking $h \to 0$ leads to vanishing magnetization at any finite N.

(d*) Now calculate the integral using the saddle-point method. To do this, expand in t the argument of the exponential

$$f(t,h) = -\frac{t^2}{2\beta J} + \ln\cosh\{\beta h + t\} + \ln 2,$$

to second order around each maximum, so that the integrand becomes a sum of two Gaussians. For these, we can do the integral analytically. When does this procedure become exact?

(e*) Repeat the calculation at finite h, and explain how the field helps spontaneous symmetry breaking in the lerge N limit. (f*) Calculate the leading $1/N$ corrections to the saddle-point approximation, for $h = 0$.

Problem 2.2 Consider a classical Ising antiferromagnet on a **"bipartite"** lattice, given by a Hamiltonian

$$H = \frac{J}{2} \sum_{<ij>} S_i S_j - h \sum_i S_i.$$

Note that now the interaction between spins minimizes the energy when the spins **anti-allign**, i.e. for $S_i = -S_j$. A bipartite lattice is one that has two sublattices, so that each spin on sublattice A interacts only with some spin on the other sublattice B. In this case, in an antiferromagnetic state, each sublattice assumes a uniform magnetization. We can introduce the magnetization for each sublattice

$$m_A = \left\langle S_i^{(A)} \right\rangle; \;\; m_B = \left\langle S_i^{(B)} \right\rangle.$$

The average magnetization then can be written as

$$m = \frac{1}{2} \left(m_A + m_B \right),$$

and the so-called "staggered" magnetization is defined by the difference between the two sublattices

$$m^\dagger = \frac{1}{2} \left(m_A - m_B \right).$$

For perfect ferromagnetic order $m = 1$, while for perfect antiferromagnetic order $m^\dagger = 1$.

(a) Use Weiss mean-field decoupling to replace one of the spins in the Hamiltonian by its thermal average. The Weiss field experienced by a given spin is then proportional to the sublattice magnetization on the other sublattice. Write down self-consistent equations for m_A and m_B, and express them through the order parameters m and $m^\dagger$.

(b) Assume that $h = 0$, so that $m = 0$, and solve the mean-field equations by expanding in $m^\dagger$. Determine the Neel (ordering) temperature, and calculate the order-parameter exponent β.

(c) Now consider a small external field $h > 0$, so that both order parameters can assume a nonzero value (Note: m will be small). By keeping only the leading terms in h and m, calculate the uniform spin susceptibility $\chi = \partial m/\partial h$, as a function of temperature. Show that χ has a **cusp** around T_N.