Solutions for HW#3

Problem 3.1

The defintion of the susceptibility is $\chi(\vec{x}) = \frac{\partial \langle S(0)\rangle}{\partial h(\vec{x})}$ where $\langle S(0)\rangle = Z^{-1}\sum_{\{S_x\}} S(0)\exp(-\beta H[S(\vec{x})])$ and the sum is taken to mean a discrete sum or integral in the continuous case. We know that within the Hamiltonian the magnetic field $h(\vec{x})$ couples to the spin, and we assume this coupling is linear. Then using the definition of χ, we find

$$\chi(\vec{x}) = \frac{\partial}{\partial h(\vec{x})}\left[Z^{-1}\sum_{\{S_x\}} S(0)\exp(-\beta H[S(\vec{x})])\right] \tag{1}$$

$$= \left(\frac{\partial Z^{-1}}{\partial h(\vec{x})}\right)\sum_{\{S_x\}} S(0)\exp(-\beta H[S(\vec{x})]) \tag{2}$$

$$+ Z^{-1}\sum_{\{S_x\}} S(0)\frac{\partial}{\partial h(\vec{x})}\exp(-\beta H[S(\vec{x})]) \tag{3}$$

$$= -Z^{-2}\sum_{\{S_x\}} \beta S(\vec{x})\exp(-\beta H[S(\vec{x})])\sum_{\{S_x\}} S(0)\exp(-\beta H[S(\vec{x})]) \tag{4}$$

$$+ Z^{-1}\sum_{\{S_x\}} S(0)(\beta S(\vec{x}))\exp(-\beta H[S(\vec{x})])\text{; From def of } \langle\bullet\rangle \text{ we get} \tag{5}$$

$$= \beta\left(\langle S(\vec{x})S(0)\rangle - \langle S(\vec{x})\rangle\langle S(0)\rangle\right) \tag{6}$$

Problem 3.2

(a) Here $\vec{k} = (k_1, k_2, \ldots, k_D)$ and $\vec{x} = (x_1, x_2, \ldots, x_D)$

$$\chi = \int \frac{d^D k}{(2\pi)^D}\frac{e^{i\vec{k}\cdot\vec{x}}}{\tilde{r}+k^2} \tag{7}$$

$$= \int \frac{d^D k}{(2\pi)^D} e^{i\vec{k}\vec{x}}\int_0^\infty \frac{dy}{2} e^{-\frac{y}{2}(\tilde{r}+k^2)} \tag{8}$$

$$= \frac{1}{(2\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dy}{2}e^{-\frac{y}{2}\tilde{r}}\int \frac{d^D k}{(2\pi)^{\frac{D}{2}}} e^{-\frac{y}{2}k^2 + ik_1x_1 + ik_2x_2 + \cdots + ik_Dx_D} \tag{9}$$

$$= \frac{1}{(2\pi)^{\frac{D}{2}}}\int \frac{1}{2}dy e^{-\frac{y}{2}\tilde{r}}\left[\frac{1}{y^{\frac{D}{2}}}e^{-\frac{1}{2}\frac{x^2}{y}}\right] \tag{10}$$

$$= \frac{1}{2(2\pi)^{\frac{D}{2}}}\int dy e^{-\frac{y}{2}\tilde{r} - \frac{1}{2}\frac{x^2}{y} - \frac{D}{2}\ln y} = \frac{1}{2(2\pi)^{\frac{D}{2}}}\int dy e^{f(y)} \tag{11}$$

In the above sequence we have used the identity

$$\int \frac{dx_1 dx_2\cdots dx_n}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}x_iA_{ij}x_j + x_iJ_i} = [DetA]^{-\frac{1}{2}}e^{\frac{1}{2}J_iA^{-1}_{ij}J_j}\text{, sumation is implied.} \tag{12}$$

Anticipating future expansions, lets pull $\tilde{r}x^2$ out of f giving $f(y) = -\frac{1}{2}\tilde{r}x^2\left(\frac{y}{x^2} + \frac{1}{\tilde{r}y} + \frac{D}{\tilde{r}x^2}\ln y\right)$ Note that $\frac{df(y)}{dy} = -\frac{1}{2}\tilde{r}x^2\left(\frac{1}{x^2} - \frac{1}{\tilde{r}y^2} + \frac{D}{\tilde{r}x^2}\frac{1}{y}\right) = 0 \to y_0 = \frac{-D\pm\sqrt{D^2+4\tilde{r}x^2}}{2\tilde{r}}$, $\frac{d^2f(y)}{dy^2} = -\frac{1}{2}\tilde{r}x^2\left(\frac{2}{\tilde{r}y^3} - \frac{D}{\tilde{r}x^2}\frac{1}{y^2}\right)$ only one solution is required since the integral over y is from zero to infinity, and can be expanded $f(y) = f_0 + \frac{1}{2}f_0''(y-y_0)^2$

$$y_0 = \frac{-D + 2x\sqrt{\tilde{r}}}{2\tilde{r}} \tag{13}$$

$$f_+(y) = -\frac{1}{2}\tilde{r}x^2\left(\frac{y}{x^2}+\frac{1}{\tilde{r}y}+\frac{D}{\tilde{r}x^2}\ln y\right) \tag{14}$$

$$-\frac{1}{4}\tilde{r}x^2\left(\frac{2}{\tilde{r}\left(\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}}\right)^3}-\frac{D}{\tilde{r}x^2}\frac{1}{\left(\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}}\right)^2}\right)(y+\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}})^2 \tag{15}$$

$$= -\frac{1}{2}\tilde{r}x^2\left(\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}x^2}+\frac{2}{-D+2x\sqrt{\tilde{r}}}+\frac{D}{\tilde{r}x^2}\ln\left(\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}}\right)\right) \tag{16}$$

$$-\frac{1}{2}\left(\frac{2\sqrt{\tilde{r}}}{x^3}\right)(y-\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}})^2 \tag{17}$$

Now, using the saddle point approximation χ becomes

$$\chi = \frac{1}{2\left(2\pi\right)^{\frac{D}{2}}}e^{-\frac{1}{2}\tilde{r}x^2\left(\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}x^2}+\frac{2}{-D+2x\sqrt{\tilde{r}}}+\frac{D}{\tilde{r}x^2}\ln\left(\frac{-D+2x\sqrt{\tilde{r}}}{2\tilde{r}}\right)\right)}\sqrt{\frac{\pi x^3}{2\tilde{r}x^2\sqrt{\tilde{r}}}} \tag{18}$$

$$= \frac{1}{2\left(2\pi\right)^{\frac{D}{2}}}e^{\frac{D-2x\sqrt{\tilde{r}}}{2}}e^{\frac{\tilde{r}x^2}{D-2x\sqrt{\tilde{r}}}}\frac{2^{\frac{D}{2}}}{\left(\frac{-D+2x\sqrt{\tilde{r}}}{\tilde{r}}\right)^{\frac{D}{2}}}r^{-\frac{3}{4}}\sqrt{\frac{\pi}{2}}\sqrt{x} \tag{19}$$

$$= \frac{1}{2\left(\pi\right)^{\frac{D}{2}}}e^{D-2x\sqrt{\tilde{r}}}e^{\frac{\tilde{r}x^2}{D-2x\sqrt{\tilde{r}}}}\frac{1}{\left(\frac{-D+2x\sqrt{\tilde{r}}}{\tilde{r}}\right)^{\frac{D}{2}}}\tilde{r}^{-\frac{3}{4}}\sqrt{\frac{\pi}{2}}\sqrt{x} \tag{20}$$

$$= \frac{1}{2\left(\pi\right)^{\frac{D}{2}}}e^{\left(\frac{D}{2}\right)}e^{\left(-\sqrt{\tilde{r}}x\right)}\frac{1}{\left(\frac{-D+2x\sqrt{\tilde{r}}}{\tilde{r}}\right)^{\frac{D}{2}}}\tilde{r}^{-\frac{3}{4}}\sqrt{\frac{\pi}{2}}\sqrt{x}\ ;\ \text{We see } \tilde{r}=\xi^{-2} \tag{21}$$

$$\chi\left(R\right) = \sqrt{\frac{\pi}{2}}\left(\frac{e}{\pi}\right)^{\frac{D}{2}}\frac{\xi^{2-D}e^{-\frac{|x|}{\xi}}}{\left(\frac{|x|}{\xi}\right)^{\frac{D}{2}-\frac{1}{2}}} \tag{22}$$

In the above sequence just about everything has been expanded e.x. f_0'' has been expanded to lowest order, etc.

(b) In the critical regime:

$$I = \int\frac{d^Dk}{\left(2\pi\right)^D}\frac{e^{i\vec{k}\cdot\vec{x}}}{k^2} \tag{23}$$

$$= \int\frac{d^Dk}{\left(2\pi\right)^D}e^{i\vec{k}\vec{x}}\int_0^\infty\frac{dy}{2}e^{-\frac{y}{2}k^2} \tag{24}$$

$$= \frac{1}{\left(2\pi\right)^{\frac{D}{2}}}\int_0^\infty\frac{dy}{2}\int\frac{d^Dk}{\left(2\pi\right)^{\frac{D}{2}}}e^{-\frac{y}{2}k^2+ik_1x_1+ik_2x_2+\cdots+ik_Dx_D} \tag{25}$$

$$= \frac{1}{\left(2\pi\right)^{\frac{D}{2}}}\int_0^\infty\frac{1}{2}dy\left[\frac{1}{y^{\frac{D}{2}}}e^{-\frac{1}{2}\frac{x^2}{y}}\right]\ ;\ \text{This integral and be done exactly for } D>2 \tag{26}$$

$$= \frac{1}{2\left(2\pi\right)^{\frac{D}{2}}}\left[2^{\frac{D}{2}-1}x^{2-D}\Gamma\left(\frac{D}{2}-1\right)\right] = \frac{1}{4\left(\pi\right)^{\frac{D}{2}}}\left[x^{2-D}\Gamma\left(\frac{D}{2}-1\right)\right]\ ;\ \Gamma \text{ is the Gamma fuction} \tag{27}$$

Therefore

$$\chi_c = \frac{1}{4\left(\pi\right)^{\frac{D}{2}}}\left[x^{2-D}\Gamma\left(\frac{D}{2}-1\right)\right] \tag{28}$$

Problem 3.3

The spin correlation function is

$$\chi(\mathbf{x}) = \langle S(0)S(\mathbf{x})\rangle = \begin{cases} \exp(-x/\xi) & x \gg \xi \\ x^{-(d-2+\eta)} & x \ll \xi \end{cases}$$

The bulk suzeptibility is given by

$$\begin{aligned} \chi &= \int \nabla^2 \mathbf{x} \chi(\mathbf{x}) \simeq \int_0^\xi \nabla^2 \mathbf{x} \chi(\mathbf{x}) \\ &= \int_0^\xi \nabla^2 x f(d) x^{d-1} x^{-(d-2+\eta)} \\ &\sim \xi^{2-\eta} \end{aligned}$$

Using the definitions for the critical exponents, $\xi \sim r^{-\nu}$ and $\chi \sim r^{-\gamma}$ close to the critical point, we get

$$\chi \sim \xi^{2-\eta} \sim r^{-\nu(2-\eta)}$$

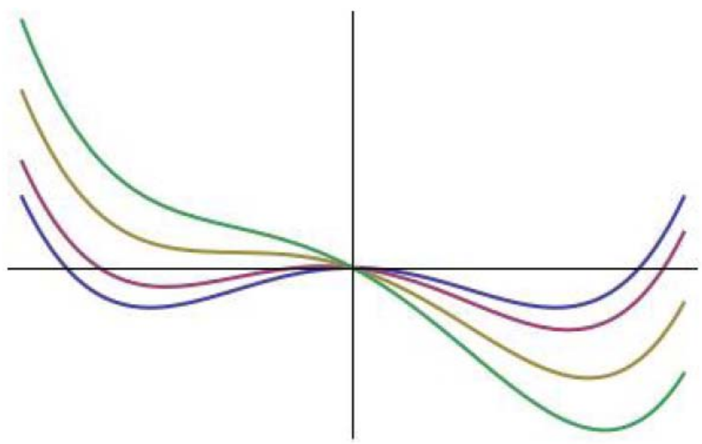
and thus

$$\gamma = \nu(2-\eta)$$

Note that this procedure is valid even if one assumes that $\chi(\mathbf{x}) = x^{d-1}x^{-(d-2+\eta)}$ ceases to be valid below $x = \xi$. In that case we can integrate only in between $x = 0$ and $x = a\xi$ where a is a number between 0 and 1 determining how long the short-range expression is valid. Changing the integral limit however does not affect the exponents, so we would still get the same result. Corrections containing the integral at large distances (from $x = a\xi$ till $x = \infty$) give subleading contributions.

Problem 3.4

(a) The potential for a ferromagnet reads $V(\phi) = \frac{1}{2}r\phi^2 + \frac{1}{4}u\phi^4 - j\phi$

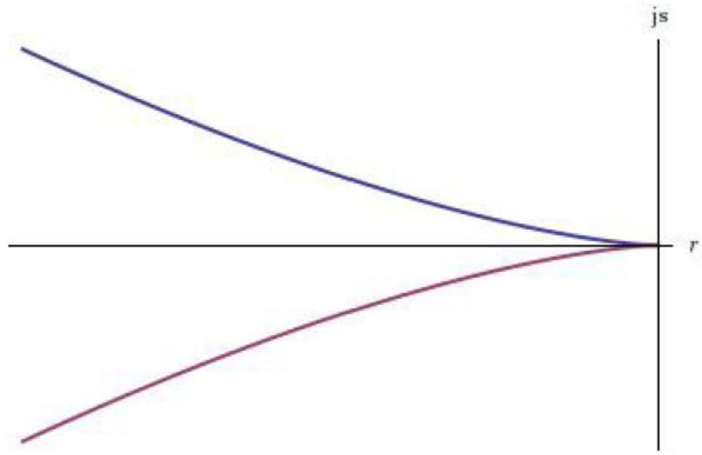


For negative r there are two minima at positive and negative magnetization ϕ. Those are defined at the solutions to $V'(\phi) = r\phi + u\phi^3 - j = 0$. For $j = 0$ we have $\phi_\pm = \pm\sqrt{|r|/u}$. For finite (but small) j those get shifted to

$$\phi_+ = \sqrt{|r|/u} + j/(2|r|) \qquad \textit{and} \qquad \phi_+ = -\sqrt{|r|/u} + j/(2|r|)$$

(b) The solution at ϕ_- gets unstable once it ceases to be a maximum, i. e. if $V''(\phi_-)$ becomes negative.

$$\begin{aligned} V''(\phi_-) &= r + 3u\phi = -|r| + 3u(|r|/u - j/\sqrt{u|r|}) = 2|r| - j\sqrt{u/|r|} \\ &\rightarrow j_s = 2|r|^{3/2}/\sqrt{u} \end{aligned}$$



The coexistence region is in between the blue and the pink line.

(c) The magnetization curve is different dependent on if we start with a positive or negative external field. Coming from large positive j we find for the magnetization:

$$\phi = \begin{cases} \phi_+(j) = \sqrt{|r|/u} + j/(2|r|) & \text{for } j > -j_s \\ \phi_-(j) = -\sqrt{|r|/u} + j/(2|r|) & \text{for } j < -j_s \end{cases}$$

While coming from large negative j we get

$$\phi = \begin{cases} \phi_+(j) = \sqrt{|r|/u} + j/(2|r|) & \text{for } j > j_s \\ \phi_-(j) = -\sqrt{|r|/u} + j/(2|r|) & \text{for } j < j_s \end{cases}$$

The magnetization is plotted below. The blue line displays the function as given above, which actually still contains the approximation of small external fields. This approximation does not necessarily hold: The coercive fields can be pretty significant. For sizeable values of j, the solutions of $\phi_\pm$ look quite nasty, but solving numerically is not a big deal. The numerical result is shown by the pink line.