Homework Assignment #3.

In the following problems, the more challenging questions are indicated by (*). These questions are required work only for the theory students.

Problem 3.1. Using general arguments from Statistical Mechanics show that the nonlocal susceptibility can be expressed as a spin-spin correlation function

$$\chi(\mathbf{x}) = \frac{\partial \langle S(0) \rangle}{\partial h(\mathbf{x})} = \langle S(0) S(\mathbf{x}) \rangle - \langle S(0) \rangle \langle S(\mathbf{x}) \rangle .$$

Problem 3.2. (a) Use the integral representation

$$\frac{1}{D} = \int_0^\infty dx \exp\{-xD\},$$

and the saddle-point method, to compute the form of the correlation function

$$\chi(R) = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{\mathbf{e}^{i\mathbf{k}\mathbf{x}}}{\widetilde{r} + k^2},$$

in the limit $R = |\mathbf{x}| \gg \xi$. Determine the correlation length ξ in terms of the Landau coefficients.

(b) Determine the exact form of $\chi = \chi_c(R)$ at the critical point ($\widetilde{r} = 0$), and show that $\chi(R) \approx \chi_c(R)$ whenever $R \ll \xi$.

Problem 3.3. Prove the following critical exponent relation

$$\gamma = (2 - \eta)\nu.$$

To do this, note that in general the spin-spin correlation function (i.e. the nonlocal susceptibility) takes the form

$$\chi(\mathbf{x}) = \langle S(0) S(\mathbf{x}) \rangle - \langle S(0) \rangle \langle S(\mathbf{x}) \rangle \sim \begin{cases} \exp\{-|\mathbf{x}|/\xi\}, & |\mathbf{x}| \gg \\ |\mathbf{x}|^{-(d-2+\eta)}, & |\mathbf{x}| \ll \xi \end{cases}.$$

[Note that $\eta = 0$ in Landau theory, but its exact value is nonzero.] The general expression for the bulk susceptibility is

$$\chi = \chi(\mathbf{k} = \mathbf{0}) = \int d\mathbf{x}\, \chi(\mathbf{x}).$$

use the fact that in the integral is dominated by the range $|\mathbf{x}| \leq \xi$ to express χ in terms of ξ, and thus relate the exponents γ, η, and ν.

Problem 3.4. Consider the system near the spinodal point using the Landau theory for an Ising ferromagnet.

(a) Determine the two uniform solutions $\phi_{\pm}(j, r)$ (stable and metastable state) as a function of j and r. .

(b) Determine the critical value of the external field$j = \pm\, j_s$ (the spinodal points), for which the metastable solutions becomes unstable. Plot the phase diagram in the $j-r$ plane, indicating the location of the spinodal lines delimiting the coexistence region.

(c) Now assume that the field is swept so quickly that the system does not have time to nucleate, and stays trapped in a metastable state, as long as it is locally stable. Imagine starting with a very large field $j = j_{\max} > j_s > 0$, such that the system is in the up-spin state $\phi_o(j) > 0$. Then imagine gradually reducing the field to $j = 0$, and then to negative values, but the system is stays in the metastable state with $\phi_o > 0$ even when $j < 0$. This persists until the "coercive field" $j = -j_s < 0$ is reached, corresponding to the negative spinodal, where the metastable state become unstable, and the magnetization reverses, and continues to decrease to more negative values as the negative field becomes stronger and stronger, down to $j = j_{\min} = -j_{\max}$. Finally, the process is reversed, and we start increasing the field again. Now the system is trapped in the $\phi_o < 0$ state until the positive spinodal is reached again. Therefore, as the field is swept up and down, the state of the system follows the hysteresis loop. Calculate the form of the hysteresis loop from LG theory, and plot it on the $\phi - j$ diagram.