Phase Transitions: Homework 5 - Solution

Problem 5.1

Starting with a gas of density n. The probability of finding a particle in infinitesimal volume element ΔV is given by $P_1(\Delta V) = n\Delta V$. This implies that the chance of finding no particle within ΔV is given by

$$P_0(\Delta V) = 1 - P_1(\Delta V) = 1 - n\Delta V$$

If a volume V is empty with a probability of $P_0(V)$ then a volume $V + \Delta V$ is empty with a chance of

$$P_0(V + \Delta V) = P_0(V)P_0(\Delta V)$$

So we find

$$\frac{\mathrm{d}P_0(V)}{\mathrm{d}V} = \frac{P_0(V + \Delta V) - P_0(V)}{\Delta V} = -n$$

Solving the differential equation with the boundary condition $P_0(0) = 1$ gives

$$P_0(V) = \exp(-nV)$$

Now, in 1D we can simply replace V by R. We know that $S(R) = S(0)$ if there has been no domain wall in between 0 and R and that if there has been a domain wall, then $S(R) = \pm S(0)$ with equal probability. I.e.:

$$S(R) = \begin{cases} S(0) & \text{with probability } p_+ = e^{-nR} + \frac{1}{2}(1 - e^{-nR}) = \frac{1}{2}(1 + e^{-nR}) \\ -S(0) & \text{with probability } p_- = \frac{1}{2}(1 - e^{-nR}) \end{cases}$$

So we find for the correlation function

$$\begin{aligned} \chi(R) = \langle S(0)S(R) \rangle &= S(0)S(0) \cdot (p_+ - p_-) \\ &= S(0)S(0) \left(\frac{1}{2}(1 + e^{-nR}) - \frac{1}{2}(1 - e^{-nR}) \right) \\ &= S(0)S(0)e^{-nR} \end{aligned}$$

Problem 5.2

$$\chi(\tau) = \mathrm{Tr}[\rho \mathbf{T}_\tau \{x(\tau)x(0)\}]$$

For $\tau > 0$ we have

$$\chi(\tau) = \sum_n \langle n|\rho e^{\tau H/\hbar} x e^{-\tau H/\hbar} x|n\rangle$$

Now use $\rho = Z^{-1}e^{-\beta H}$ to show that

$$\begin{aligned}
\sum_n \rho|n\rangle &= \left(\sum_m e^{-\beta E_m}\right)^{-1} (e^{-\beta E_0}|0\rangle + e^{-\beta E_1}|1\rangle + \ldots) \\
&= \left(\sum_m e^{-\beta E_m}\right)^{-1} e^{-\beta E_0} \left(|0\rangle + e^{-\beta(E_1-E_0)}|1\rangle + \ldots\right) \\
&\overset{\beta\to\infty}{=} e^{-\beta E_0}|0\rangle
\end{aligned}$$

So we find

$$\begin{aligned}
\chi(\tau) &= e^{-\beta E_0}\langle 0|e^{\tau H/\hbar} x e^{-\tau H/\hbar} x|0\rangle \\
&= \sum_{n,m} e^{-\beta E_0} e^{\tau E_0/\hbar} \langle 0|x|n\rangle\langle n|e^{-\tau H/\hbar}|m\rangle\langle m|x|0\rangle \\
&= \sum_n e^{-\beta E_0} e^{-\tau(E_n - E_0)/\hbar} \langle 0|x|n\rangle\langle n|x|0\rangle \\
&= \sum_n e^{-\beta E_0} |\langle 0|x|n\rangle|^2 e^{-\tau(E_n-E_0)/\hbar}
\end{aligned}$$

Because the potential is symmetric, we know that the energy eigenstates $|0\rangle$, $|1\rangle$ etc must be of definite parity ± 1. Because x is of negative parity $\langle 0|x|1\rangle = 0$. Consider now a system where just the ground and the first excited states have to be considered, i.e. the excitation energy to the second excited state $\Delta_2 = E_2 - E_0$ is large enough compared to $\tau/\hbar$ and $\Delta = E_1 - E_0$ such that $e^{-\tau\Delta/\hbar} \gg e^{-\tau\Delta_2/\hbar}$.

$$\begin{aligned}
\chi(\tau) &= \sum_n e^{-\beta E_0} |\langle 0|x|n\rangle|^2 e^{-\tau(E_n-E_0)/\hbar} \\
&= e^{-\beta E_0}|\langle 0|x|1\rangle|^2 \cdot e^{-\tau\Delta/\hbar} + e^{-\beta E_0}|\langle 0|x|2\rangle|^2 \cdot e^{-\tau\Delta_2/\hbar} + \ldots \\
&\sim e^{-\tau\Delta/\hbar}
\end{aligned}$$