Homework Assignment #5.

Problem 5.1. Consider a 1D Ising model, where each spin configuration can be represented by a corresponding configuration of domain walls. At sufficiently low temperatures, these domain walls are very dilute, and thus can be considered to be a gas of noninteracting "particles" with uncorrelated random positions and average density $n = e^{-\mu/T}$, with μ being the energy of a single domain wall. Calculate the average correlation function $\chi(R) = \langle S(0)S(R)\rangle$ by averaging over the (random) positions of domain walls, and show that

$$\chi(R) \sim \exp\{-R/\xi\},$$

where the correlation length $\xi \sim n^{-1}$. [**Hint:** Physically, the correlation function measures the probability that the spin $S(R)$ has the same orientation as $S(0)$. This will be possible if there are *no domain walls at all* in a region of size R. Since the domain walls are randomly positioned, there occasionally will be found such large "cavities", but this will occur with a very small probability. The corresponding probability is given by the Poisson distribution. Derive the probability of finding a "cavity" of diameter R in arbitrary dimensions in a gas of uncorrelated particles of a given density n. By using this result in $d = 1$ prove the desired theorem.

Problem 5.2. Consider the imaginary time (Matsubara) auto-correlation function

$$\chi(\tau) = Tr[\rho \mathbf{T}_\tau \{x(\tau)x(0)\}],$$

for a quantum particle. Assuming that the system is described by a Hamiltonian $\mathbf{H}$ with known eigenvalues E_n, and eigenvectors $|n\rangle$ $(n = 0, 1, ...)$, evaluate this correlation function at low temperatures and show that

$$\chi(\tau) \sim e^{\tau\Delta/\hbar},$$

with $\Delta = E_1 - E_o$ being the splitting between the ground state and the first excited state.