Homework Assignment #7.

In the following problems, the more technical questions are indicated by (*). These questions should be solved only by theory students.

Problem 7.1

(a) The Landau potential for an antiferromagnet in a uniform external field takes the general form:

$$V(\phi, \phi^{\dagger}, h) = \frac{1}{2} r \left(\phi^{\dagger}\right)^2 + \frac{1}{4} u \left(\phi^{\dagger}\right)^4 + \frac{1}{6} v \left(\phi^{\dagger}\right)^6 + \frac{1}{2} r_1 \phi^2 - j\phi + w \left(\phi^{\dagger}\right)^2 \phi^2.$$

Show that for $j = 0$, $u > 0$, and $v > 0$ the transition to the antiferromagnetic state is of second order, but for $u < 0$ it is of first order.

(b) Now consider finite external field $j > 0$, but $u > 0$. Show that for small enough field $j < j_c$ the transition is second order, but for larger field the transition becomes of first order. Determine the value of the "tricritical" field j_c from the Weiss theory. Construct the phase diagram in the $j - T$ plane. Determine the critical exponents **at the tricritical point**.

Problem 7.2 Consider the one dimensional Ising model, and use the real-space RG approach to compute the leading low temperature behavior of the specific heat at $h = 0$. To do this, follow the procedure similar as that we used for the correlation length, but this time use the Kadanoff scaling relation for the free energy

$$f(t, h) = b^{-d} f(K(b), h(b)).$$

Problem 7.3 Consider the Ising model on a hypercubic lattice in arbitrary dimension $d > 1$. Show that the Migdal-Kadanoff bond-moving approximation discussed in class can be generalized to produce the following recursion relations

$$K' = \frac{1}{2} \ln \cosh(2dK),$$
$$h' = h \left[1 + \tanh(2dK)\right].$$

(a) Consider the d as a continuous parameter to perform a formal expansion around $d = 1$. To do this, write $d = 1 + \varepsilon$, and consider $\varepsilon \ll 1$. Show that in this case the fixed point

$$K^* \sim 1/\varepsilon,$$

and calculate its value to leading order in ε [The critical point is still described by $h^* = 0$.] As we can see from this calculation, the finite temperature transition exists whenever $d = 1 + \varepsilon > 1$. Thus, $d = 1$ is indeed the **lower critical dimension** for the Ising model.

(b) Expand the recursion relations around this fixed point (i.e. expand the expression for K' to leading order in $t = K - K^*$), and compute the exponents λ_t and λ_h to leading order for $\varepsilon \ll 1$.

(c) Use the Kadanoff theory to derive expressions for all the six critical exponents α, β, γ, δ, η, and ν, in terms of λ_t and λ_h. The use the above ε-expansion results for λ_t and λ_h to obtain expressions for all the exponents.

(d) Use these expressions to obtain an estimate for the critical exponents in $d = 2$ and $d = 3$. Compare your results to the known exact results (you can find these values in one of the textbooks). How accurate are the approximate RG calculations we have used?

(e*) Show that the Migdal-Kadanoff approximation is exact on the hierarchical lattice described in class (see lecture notes). Calculate the **fractal dimension** d_f of this lattice (to do this you may do a bit of literature search on the properties of hierarchical lattices; a good starting point is the paper quoted in the lecture notes). Calculate all the critical exponent for this lattice. Compare your results to what you would obtain by using the above $1 + \varepsilon$ expansion by coo sing $\varepsilon = d_f$.