Phase Transitions - Homework 7 - Solutions

PROBLEM 7.2

(a) In 1 dimension, the free energy (per unit cell) is given by

$$f(t,h) = b^{-1} f(K(b), h(b)),$$

where b is the scaling factor by which the lattice spacing got increased, h is the external field and K is the effective interaction strength $K = J/T$. In the 1 dimensional case, where $T_c = 0$, so we are dealing with a critical point at $K^* \to \infty$. From decimation, we find

$$K' = \frac{1}{2} \ln \cosh(2K)$$
$$h' = h(1 + \tanh(2K))$$

In the environment of the critical point, at $k \gg 1$, this means

$$K' \approx \frac{1}{2} \ln(\frac{1}{2} e^{2K}) = K - \frac{1}{2} \ln 2$$
$$h' \approx h(1+1) = 2h$$

In each decimation step, d$b = 2$, so

$$\mathrm{d}K = -\frac{1}{2} \ln 2 \quad \to \quad \beta_K = \frac{\mathrm{d}K}{\mathrm{d}\ln b} = -\frac{1}{2}$$
$$\mathrm{d}h = h \quad \to \quad \beta_h = \frac{\mathrm{d}h}{\mathrm{d}\ln b} = \frac{h}{\ln 2}$$

Each decimation steps drives us further away from the critical point towards $K(b) = 0$, where from $K(b) = K_0 - \frac{1}{2} \ln b$ we get

$$b = e^{2K_0}$$

At this point

$$h(b) = 2b \cdot h = 2e^{2K_0} h$$

So we get

$$f(t,h) = b^{-1} f(K(b), h(b)) = e^{-2K_0} f\left(0, h \cdot 2e^{2K_0}\right) = e^{-2J/T} f\left(h \cdot 2e^{2J/T}\right)$$

With this we can calculate:

$$C = -\frac{\partial^2 f(0)}{\partial T^2} \propto -\frac{\partial^2 e^{-2J/T}}{\partial T^2} = \frac{2J}{T^3}(1 + \frac{2J}{T})e^{-2J/T}$$

So at low temperatures we have

$$C \propto T^{-4}e^{-2J/T}$$

As the exponential goes faster to 0 than any power law, $C \to 0$ at $T \to 0$.

(b) For $h \neq 0$ we obtain the low temperature dependence of the magnetization m and the susceptibility χ by taking derivatives (**??**) w. r. t. h.

$$m = \left.\frac{\partial f(t,h)}{\partial h}\right|_{h=0} \sim \frac{\partial}{\partial h} e^{-2J/T} f\left(h \cdot 2e^{2J/T}\right) = \text{const.}$$
$$\chi = \left.\frac{\partial^2 f(t,h)}{\partial h^2}\right|_{h=0} \sim e^{2J/T} \qquad \text{at } T \to 0,\ \chi \text{ blows up}$$

PROBLEM 7.3

As discussed in class decimation only truly works in $d = 1$, because each decimation step would create interactions between more and more particles, rendering each further step more and more involved until reaching unsolvability. However decimation can be applied when moving the bonds between the individual sites by the following scheme:

Considering N sites in an hypercubical lattice, in d dimensions, on has $d \cdot N$ bonds total. Now you reduce the number of sites to $M = N/(d+1)$[**?**] which means you have $d \cdot N = d \cdot (d+1) \cdot M$ bonds so each of the new sites is attached to $d \cdot 2^d$ bonds. Now dividing that by M times the average coordination number $d+1$ (the average of $2d$, which is the coordination number of the edges of the new cube, and 2, the coordination number of the sites along the sides of the cube) you have d bonds. What this amounts to is that the bond strength increased: from K before decimation to $d \cdot K$ after decimation.

From this point on it is very easy to get rid of half of the sites in the new lattice by decimation. Just sum over the possible spin configurations of the sites at the sides of the cube setting the interaction strength to $d \cdot K$ and obtain:

$$K^{'} = \frac{1}{2} \ln \cosh(2dK)$$
$$h^{'} = h[1 + \tanh(2dK)]$$

(a) Setting $d = 1 + \varepsilon$ and expanding in small ε, we find

$$K' - \frac{1}{2}\ln\cosh(2dK) - \varepsilon K\tanh(2K) + \mathcal{O}(\varepsilon^2) = 0$$

Thus the fixpoint is found by solving the above expression for $K' = K = K^*$. Using that $\cosh(2K^*) \to \frac{1}{2}e^{2K^*}$ and $\tanh(K^*) \to 1$ for $K^* \to \infty$ we find that

$$K^* - (K^* - \frac{1}{2}\ln 2 + \varepsilon K^*) = 0 \qquad \text{for} \quad K^* = \frac{1}{2\varepsilon}\cdot\ln 2$$

(b) Computation of the RG exponents:

$$K = K^* + t \qquad \text{where } t = K - K^*$$

$$K' = \frac{1}{2}\ln\cosh(2d(K^* + t))$$

expanding around small t

$$K' = \frac{1}{2}\ln\cosh(2dK^*) + d\cdot t\tanh(2dK^*) = K^* + d\tanh(2dK^*)\cdot t$$

using $K^* \sim 1/\varepsilon$ and expanding around small ε

$$K' = K^* + (1+\varepsilon) * t = K^* + t'$$

As by decimation we increased the lattice spacing by a factor $b = 2$, the RG approach for t is $t' = b^{\lambda_t}\cdot t = 2^{\lambda_t}\cdot t$. We can read off:

$$\lambda_t = \frac{\varepsilon}{\ln 2} + \mathcal{O}(\varepsilon^2)$$

For the field strength we find

$$h' = 2^{\lambda_h} h \qquad \text{(RG ansatz)}$$

$$= h[1 + \tanh(2dK^*)] = 2h + \mathcal{O}(\varepsilon)$$

So

$$\lambda_h = 1 + \mathcal{O}(\varepsilon)$$

(c) Using $f(t,h) = t^{d/\lambda_t} f\left(1, \frac{h}{t^{\lambda_h/\lambda_t}}\right)$ and $t \sim T - T_c$

- **correlation length** $\xi \sim |t|^{-\nu}$
  $\xi(t,h) = b\cdot\xi(b^{\lambda_t}t, b^{\lambda_h}h) = t^{-1/\lambda_t}\xi\left(1, \frac{h}{t^{\lambda_h/\lambda_t}}\right)$ so $\xi(t,0) \sim t^{-1/\lambda_t}$ and so

$$\nu = 1/\lambda_t$$

- **specific heat** $C \sim |t|^{-\alpha}$

  $C = -\frac{\partial^2 f(t,h)}{\partial T^2} \sim t^{t/\lambda_t - 2}$ and so

$$\alpha = 2 - \frac{d}{\lambda_t}$$

- **magnetization** $m(h = 0) \sim |t|^{\beta}$

  $m = \frac{\partial f(t,h)}{\partial h}\Big|_{h=0} \sim t^{(d-\lambda_h)/\lambda_t}$ and so

$$\beta = \frac{d - \lambda_h}{\lambda_t}$$

- **magnetization at the critical point** $m_c(h) \sim h^{1/\delta}$

  $m_c(h) = t^{\beta} f'\left(1, \frac{h}{t^{\lambda_h/\lambda_t}}\right) = t^{\beta} f'(y)$ where $y = \frac{h}{t^{\lambda_h/\lambda_t}}$. To obtain a finite value for $t \to \infty$ we need $f'(y) \sim t^{-\beta}$. So it follows $f'(y) \sim y^{\beta\lambda_t/\lambda_h}$ and $m_c(h) \sim h^{\beta\lambda_t/\lambda_h}$ and so

$$\delta = \lambda_h/\beta\lambda_t = \lambda_h/(d - \lambda_h)$$

- **suszeptibility** $\chi \sim |t|^{-\gamma}$

  $\chi = \frac{\partial^2 f(t,h)}{\partial h^2}\Big|_{h=0} \sim t^{(d-2\lambda_h)/\lambda_t}$ and so

$$\gamma = \frac{2\lambda_h - d}{\lambda_t}$$

- **correlation function at the critical point** $\chi_c \sim \xi^{2-\eta}$

  $\chi_c \sim t^{-\gamma} \sim \xi^{\gamma/\nu}$ and so we reproduces Fisher's scaling law and obtain

$$\eta = 2 - \gamma/\nu = 2 - 2\lambda_h + d$$

(d) Critical exponents for $d = 2$ (implying $\varepsilon = 1$, $\lambda_t = \frac{1}{\ln 2}$, $\lambda_h = 1$):

$$\nu = \ln 2$$

$$\alpha = 2 \cdot (1 - \ln 2) = 0.614$$

$$\beta = \ln 2 = 0.693$$

$$\delta = 1$$

$$\gamma = 0$$

$$\eta = 2$$

Critical exponents for $d = 3$ (implying $\varepsilon = 2$, $\lambda_t = \frac{2}{\ln 2}$, $\lambda_h = 1$):

$$\begin{aligned} \nu &= \frac{1}{2}\ln 2 = 0.347 \\ \alpha &= 2 - \frac{3}{2}\ln 2 = 0.960 \\ \beta &= \ln 2 = 0.693 \\ \delta &= \frac{1}{2} \\ \gamma &= -\frac{1}{2}\ln 2 = -0.347 \\ \eta &= 3 \end{aligned}$$

This results, however, should not be trusted too much, since $\varepsilon = 1, 2$ is in contradiction to our initial assumption $\varepsilon \ll 1$.

(e) The Migdal-Kadanoff approximation is exact on the diamond hierarchical lattice. This can be explained by the iterative prescription how to build up a lattice, which goes as follows:

1. Between two points, draw a line with length L
2. Add two points to the sides of the line, such that their distance from the both the original points is $x \cdot L$. For this, it is necessary that $x > \frac{1}{2}$
3. Connect each of the new points which each of the original points (but not among themselves)
4. Erase the original line
5. Repeat this steps for all the new lines individually. The number x characterize the lattice (it determines the opening angle of the 'diamond' $\cos(\frac{\alpha}{2}) = \frac{1}{2x}$).

Now, we can do decimation by going through this iterative procedure backwards. As the sites which where added in the last iteration step are only connected to two other sites, we can directly sum over their spins, without having to move bonds. Therefore, as in 1 dimension, the Migdal-Kadanoff approach is exact.
The fractional dimension of the lattice is defined[**?**] as

$$D_f = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(\frac{1}{\varepsilon})}$$

where ε is the length of the pieces constituing the fractal and $N(\varepsilon)$ the number of pieces. So for the diamond lattice we find

$$D_f = \lim_{k\to\infty} \frac{\ln 4^k}{\ln \frac{1}{x^k}} = \frac{\ln 4}{\ln \frac{1}{x}}$$

The constraint $x > \frac{1}{2}$ requires $D_f > 2$ and in order for neighbouring diamonds not to overlap we also need $x < \frac{1}{\sqrt{2}}$ which renders $D_f < 4$. Such, the fractional dimension of the diamond lattice depends on x, but is generally a number between 2 and 4. This is the result one would expect as the dimension of a fractal has to be larger or equal than its topological dimension (in this case 2) and smaller or equal to its Hamel dimension (in this case 4).

[1] 2^d hypercubes of with sidelength L make a big hypercube with sidelength $2 \cdot L$. Probably one would guess at first that $M = N/2^d$, but we are still at bond-moving. Then after that step, decimation gets rid of more sites, leaving $N/2^d$ after bond-moving and decimation. In other words, you disconnect a certain number of sites by moving bonds away from them and making them not to interact at all. The number $M = N/(d+1)$ can be derived by counting the number of edges and sides of a hypercube.

[2] Haussdorff dimension of a fractal