Phase Transitions - Homework 8: Solutions

Problem 8.1

(a) Any real symmetric matrix K can be diagonalized by a similarity transformation $\tilde{K} = A^{-1}TA = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where A is the transformation matrix and the λ_i are the eigenvalues of K. Defining the field and spin vectors in the new basis $\tilde{S} = AS$ and $\tilde{\phi} = A\phi$ we can write:

$$Z = \int \prod_{i=1}^{n} \mathrm{d}\phi_i \exp\left[-\frac{1}{2}\phi^T K\phi + S^T K\phi\right]$$

$$= \int \prod_{j=1}^{n} J^{-1}\mathrm{d}\tilde{\phi}_j \exp\left[-\frac{1}{2}\tilde{\phi}^T \tilde{K}\tilde{\phi} + \tilde{S}^T \tilde{K}\tilde{\phi}\right]$$

where J is the Jacobian of the basis transformation

$$= \prod_{j=1}^{n} J^{-1}\left(\int \mathrm{d}\tilde{\phi}_j \exp\left[-\frac{1}{2}\lambda_j\tilde{\phi}_j^2 + \tilde{S}_j\lambda_j\tilde{\phi}_j\right]\right)$$

now we can use the Gaussian identities to obtain

$$= \prod_{j=1}^{n} J^{-1}\left(\frac{2\pi}{\lambda_i}\right)\exp\left[\frac{\lambda_i S_i^2}{2}\right]$$

$$\propto \exp\left[\frac{1}{2}S^T K S\right]$$

For the partition function, the prefactors can be completely ignored as the just give additive constants under the Log.

(b)

$$S[\phi] = \int \mathrm{d}x \frac{1}{2}\phi^T K\phi - \ln\cosh\left(K\phi + h\right)$$

then expand around small fields

$$\approx \int \mathrm{d}x \frac{1}{2}K\phi^2 - \ln\cosh\left(K\phi\right) - h\tanh\left(K\phi\right)$$

and also expand in small fields

$$\approx \int \mathrm{d}x \frac{1}{2}K\phi^2 - \frac{1}{2}K^2\phi^2 + \frac{1}{12}K^4\phi^4 - hK\phi$$

$$= K\int \mathrm{d}x \left[\frac{1}{2}(1-K)\phi^2 + \frac{1}{12}\phi^4 K^3 - h\phi\right]$$

Again, the overall prefactor is not important. Going into momentum space $\phi(x) \to \int \frac{\mathrm{d}^d k}{(2\pi)^d}\phi(k)$ we can write $K(k) = f_K \sum_i^d \cos(k_i a)$.
In a hypercubic lattice with coordination number $z = 2d$ for symmetry reasons all k_i are equal. We are only interested in the long-ranged interactions and therefore expand in small momenta
$K(k) \approx \frac{1}{2} f_K z[1 - \frac{1}{2}a^2k^2 + \mathcal{O}(k^4)] \to K(x) = Kz * (1 + \nabla^2)$.
Plugging that in we get

$$S[\phi] \sim \int \mathrm{d}x \left(\frac{1}{2}\phi(x)[1 - Kz - \nabla^2]\phi(x) + \frac{1}{4}\phi(x)^4 \frac{Kz}{3} - h(x)\phi(x) \right)$$

Comparing with the Landau form and using $K_c = z$ one reads off:

$$r = 1 - Kz = (K_c - K)/K_c \sim (T - Tc)$$

and

$$u = \frac{Kz}{3} = \frac{K}{3K_c} \approx \frac{1}{3} \text{ close to the critical point}$$

Problem 8.2

(a) Requiring that under rescaling $x \to xb$ the operator $\frac{1}{2}\int \mathrm{d}^d x \nabla^2 \phi^2$ remains constant we found in class that $\phi \to b^{1-d/2}\phi$. From that, it is easy to see that

$$w_m \int \mathrm{d}^d x \phi^m(x) \to w_m \int \mathrm{d}^d b^d x b^{m(1-d/2)} \phi^m$$

Absorbing the rescaling into the coefficient w_m therefore means
$w_m \to w_m \cdot b^{d+m(1-d/2)}$ so

$$\lambda_w = m - (m-2)d/2$$

(b) We found in class that the field strength in terms of the parameters r, j, u, $w\ldots$ behaves under scaling as

$$\phi(r, u, j, w, \ldots) = b^{-d+\lambda_j}\phi\left(b^{\lambda_r} r, b^{\lambda_j} j, b^{\lambda_u} u, b^{\lambda_w} w, \ldots\right)$$

We want to figure out the critical exponent δ, determined by the behavior of ϕ at the critical point $(r \to 0)$ $\phi_c \sim j^{1/\delta}$ Naively, one would choose $b^{\lambda_j} j = 1$, i.e. $b = j^{-1/\lambda_j}$ and from

$$\phi(r, u, j = 1, w, \ldots) = j^{(d-\lambda_j)/\lambda_j}\phi(0, 0, 1, 0 \ldots)$$

and end of with $\delta = \lambda_j/(d-\lambda_j) = \frac{d+2}{d-2}$.
However, consider the free energy:

$$F(r,u,j,w\ldots) = \frac{1}{2}r\phi^2 + \frac{1}{4}u\phi^4 - j\phi + \mathcal{O}(\phi^w).$$

So at the critical point $r=0$ the minimum is found at

$$\phi \approx \left(\frac{j}{u}\right)^{1/3}$$

So in fact,

$$\phi(0,1,u(b)\to 0,0,\ldots) \sim u^{-1/3} \sim b^{-\lambda_u/3} \sim j^{\lambda_u/(3\lambda_j)}$$

So actually

$$\phi_c = j^{(d-\lambda_j)/\lambda_j + \lambda_u/(3\lambda_j)}$$

and this gives

$$\delta = \frac{3\lambda_j}{3\lambda_j} 3(d-\lambda_j) + \lambda_u$$

With $\lambda_j = 1 + d/2$ and $\lambda_u = 4-d$ this gives

$$\delta = 3$$