Homework Assignment #9.

ε-Expansions and Symmetry Breaking Perturbations

In the following problems, the more challenging questions are indicated by (*). These questions should be answered only by the theory students.

Problem 1. Derive the RG equations corresponding to an $O(n)$ version of the ϕ^4-theory using the ε-expansion approach and momentum-shell method of Wilson and Fisher. Calculate the values of all the exponents as a function of n and d.

Problem 2*. Consider a perturbation breaking the $O(n)$ symmetry (describing the single ion anisotropy), of the form

$$\delta S_g = \frac{1}{2} g \int d\mathbf{x}\ \phi_n^2(\mathbf{x}).$$

Here, $\phi_n(\mathbf{x})$ is the n-th component of the vector field $\phi_\alpha(\mathbf{x})$ ($\alpha = 1, ..., n$) . By separating the components $\alpha = 1, ..., n-1$ and $\alpha = n$, action then can be written as

$$S[\phi] = \frac{1}{2}\sum_{\alpha=1}^{n-1} \int d\mathbf{x}\, \phi_\alpha(\mathbf{x})\, [r - \nabla^2]\, \phi_\alpha(\mathbf{x}) + \frac{1}{2} \int d\mathbf{x}\, \phi_n(\mathbf{x})\, [r + g - \nabla^2]\, \phi_n(\mathbf{x}) +$$

$$+ \frac{u}{4} \int d\mathbf{x} \sum_{\alpha,\beta=1}^{n-1} \phi_\alpha^2(\mathbf{x})\phi_\beta^2(\mathbf{x}) + \frac{v}{2} \int d\mathbf{x} \sum_{\alpha=1}^{n-1} \phi_\alpha^2(\mathbf{x})\phi_n^2(\mathbf{x}) + \frac{w}{4} \int d\mathbf{x}\ \phi_n^2(\mathbf{x})\phi_n^2(\mathbf{x}).$$

Note that the propagator of the $\alpha = n$ components now has "mass" $r+g$, while all the other components still have "mass" r. Diagrammatically, we need to consider diagrams with two different types of propagators. Note that the "bare" values of the interaction amplitudes u, v, and w are all the same and equal to u. However, the three terms differ by symmetry, since in presence of the g-perturbation the channel $\alpha = n$ is different from the other channels. Therefore, we expect the three terms to renormalize differently under RG, and we thus consider three different types of vertices as well. To simplify the notation, also rescale the interaction amplitude to absorb the factors Ω_d that arise from momentum shell integration.

Derive nonlinear RG equations in presence of this perturbation, and show that

$$
\begin{aligned}
\frac{dr}{d\ell} &= 2r + (n+1)\frac{u}{r+\Lambda^2} + \frac{v}{r+g+\Lambda^2};\\
\frac{d(r+g)}{d\ell} &= 2(r+g) + (n-1)\frac{v}{r+\Lambda^2} + 3\frac{w}{r+g+\Lambda^2};\\
\frac{du}{d\ell} &= \varepsilon u - (n+7)\frac{u^2}{\left(r+\Lambda^2\right)^2} - \frac{v^2}{\left(r+\Lambda^2\right)^2};\\
\frac{dv}{d\ell} &= \varepsilon v - (n+1)\frac{uv}{\left(r+\Lambda^2\right)^2} - 3\frac{vw}{\left(r+g+\Lambda^2\right)^2} - 4\frac{v^2}{\left(r+g+\Lambda^2\right)\left(r+\Lambda^2\right)};\\
\frac{dw}{d\ell} &= \varepsilon w - (n-1)\frac{v^2}{\left(r+\Lambda^2\right)^2} - 9\frac{w^2}{\left(r+g+\Lambda^2\right)^2}.
\end{aligned}
$$