Phase Transitions - Homework 9

Problem 9.1

In ϕ^4 theory, the following diagrams contribute to the renormalization:

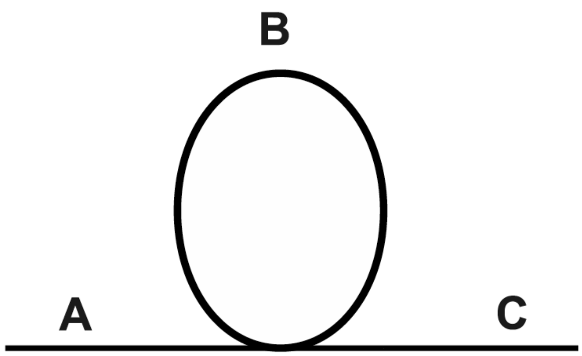

Figure 1: One loop contribution to the 2-point-function of ϕ^4 theory, affecting the renormalization of r. A, B, C are labels for the respective fields.

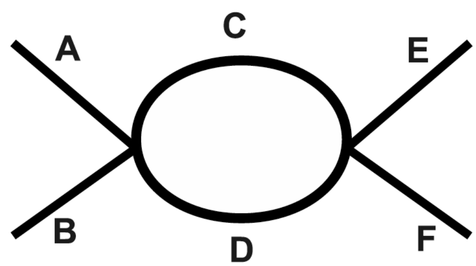

Figure 2: One loop contribution to the 4-point-function of ϕ^4 theory, affecting the renormalization of u. A, B, C, D, E, F are labels for the respective fields.

In ϕ^4 theory to $\mathcal{O}(n)$, the action reads:

$$S[\phi_{1,2,\dots n}] = \frac{1}{2}\sum_{\alpha=1}^{n}\int \mathrm{d}\mathbf{x}\phi_\alpha(\mathbf{x})[r-\nabla^2]\phi_\alpha(\mathbf{x}) + \frac{u}{4}\int \mathrm{d}\mathbf{x}\sum_{\alpha,\beta=1}^{n}\phi_\alpha^2(\mathbf{x})\phi_\beta^2(\mathbf{x}) \quad (1)$$

Let us list all the diagrams that contribute to the renormalization of r:

A	B	C	number of diagrams	multiplicity
α	α	α	1	3
α	β	α	$n-1$	1

In class, we went through the calculation for ϕ^4 theory of one field, so only the first process of the above table was present. We found

$$n = 1: \qquad \beta_r = 2r + 3\Omega_d \frac{u}{r+\Lambda^2}$$

where the first term reflects just the scaling of the operator itself, while the second term contains the contribution from the 1 loop diagram. In ϕ^4 theory to $\mathcal{O}(n)$, we have to add the processes from the second line and get

$$\text{general } n: \qquad \beta_r = 2r + (n+2)\Omega_d \frac{u}{r+\Lambda^2}$$

The diagrams that contributing to the renormalization of u are:

	A	B	C	D	E	F	number of diagrams	multiplicity
ϕ_α^4 interactions	α	α	α	α	α	α	1	9
	α	α	β	β	α	α	$n-1$	1
$\phi_\alpha^2\phi_\beta^2$ interactions	α	α	α	α	β	β	1	3
	α	α	β	β	β	β	1	3
	α	β	α	β	α	β	4	1
	α	α	γ	γ	β	β	$n-2$	1

We see that in total the $n+8$ diagrams contribute to the interaction of 4 identical ϕ as well as to the interaction of two pairs of different fields. This justifies the introduction of one common u in equ. (**??**). Again comparing with the $n=1$ solved in class

$$n=1: \qquad \beta_u = (4-d)u - 9\Omega_d \frac{u^2}{(r+\Lambda^2)^2}$$

we get for the general case

$$\text{general } n: \qquad \beta_u = (4-d)u - (n+8)\Omega_d \frac{u^2}{(r+\Lambda^2)^2}$$

As we did in the lecture, let us intrduce $\varepsilon = 4-d$, $a = \frac{3\Omega_d}{\Lambda^2}$ and $b = \frac{3\Omega_d}{\Lambda^4}$. The beta-functions then read

$$\beta_r = 2r + (n+2)\Omega_d \frac{u}{r+\Lambda^2} \sim 2r + \frac{n+2}{3}au - \frac{n+2}{3}bur$$
$$\beta_u = (4-d)u - 9\Omega_d \frac{u^2}{(r+\Lambda^2)^2} \sim \varepsilon u - \frac{n+8}{3}bu^2$$

We find the Gaussian fixed point at $r_0^\star = u_0^\star = 0$. Expanding around that fixed point gives:

$$\begin{pmatrix} \frac{\mathrm{d}r}{\mathrm{d}l} \\ \frac{\mathrm{d}u}{\mathrm{d}l} \end{pmatrix} = \begin{pmatrix} 2 & \frac{n+2}{3}a \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}$$

and we find the exact same exponents as for the $n=1$ case:

$$\lambda_1^0 = 2 \qquad \text{and} \qquad \lambda_2^0 = \varepsilon$$

The Wilson-Fisher fixed point is found at

$$u^\star_{\rm WF} = \frac{3\varepsilon}{(n+8)b} \qquad \text{and} \qquad r^\star_{\rm WF} = -\frac{n+2}{2(n+8)}\frac{a}{b}\varepsilon$$

Expanding around the WF fixed point gives:

$$\begin{pmatrix} \frac{\mathrm{d}r}{\mathrm{d}l} \\ \frac{\mathrm{d}u}{\mathrm{d}l} \end{pmatrix} = \begin{pmatrix} 2 - \frac{n+2}{n+8}\varepsilon & \frac{n+2}{3}a + \frac{(n+2)^2}{6(n+8)}a\varepsilon \\ 0 & -\varepsilon \end{pmatrix} \begin{pmatrix} \delta r \\ \delta u \end{pmatrix}$$

and we find the following exponents:

$$\lambda_1^{\rm WF} = 2 - \frac{n+2}{n+8}\varepsilon \qquad \text{and} \qquad \lambda_2^{\rm WF} = -\varepsilon$$

Problem 9.2

The action

$$S[\phi_{1,2,\ldots n}] = \frac{1}{2}\sum_{\alpha=1}^{n-1}\int \mathrm{d}\mathbf{x}\phi_\alpha(\mathbf{x})[r-\nabla^2]\phi_\alpha(\mathbf{x}) + \frac{1}{2}\int \mathrm{d}\mathbf{x}\phi_n(\mathbf{x})[r+g-\nabla^2]\phi_n(\mathbf{x})$$
$$+\frac{u}{4}\int \mathrm{d}\mathbf{x}\sum_{\alpha,\beta=1}^{n-1}\phi_\alpha^2(\mathbf{x})\phi_\beta^2(\mathbf{x}) + \frac{v}{2}\int \mathrm{d}\mathbf{x}\sum_{\alpha=1}^{n-1}\phi_\alpha^2(\mathbf{x})\phi_n^2(\mathbf{x}) + \frac{w}{4}\int \mathrm{d}\mathbf{x}\phi_n^4(\mathbf{x})$$

describes the dynamics of $n-1$ fields of mass r, labelled by α, β and one field ϕ_n of mass $r+g$. There exist three types of ϕ^4 couplings:

- proportional to coupling constant u: connecting 4 light fields ϕ_α^4 or $\phi_\alpha^2\phi_\beta^2$ (As before, introducing a common u will be justified by the common RG running)
- proportional to coupling constant u: connecting 2 light fields and 2 heavy fields $\phi_\alpha^2\phi_n^2$
- proportional to coupling constant u: connecting 4 heavy fields ϕ_n^4

Puzzling together possible diagrams for the renormalization of r and $r+g$ we find:

	A	B	C	coupling	propagators	diagrams	multiplicity
r	α	α	α	u	$(r+\Lambda^2)^{-1}$	1	3
	α	β	α	u	$(r+\Lambda^2)^{-1}$	$n-2$	1
	α	N	α	v	$(r+g+\Lambda^2)^{-1}$	1	3
$r+g$	N	α	N	v	$(r+\Lambda^2)^{-1}$	$n-1$	1
	N	N	N	w	$(r+g+\Lambda^2)^{-1}$	1	3

So we get:

$$\frac{\mathrm{d}r}{\mathrm{d}l} = 2r + (n+1)\frac{u}{r+\Lambda^2} + \frac{v}{r+g+\Lambda^2}$$
$$\frac{\mathrm{d}(r+g)}{\mathrm{d}l} = 2(r+g) + (n-1)\frac{v}{r+\Lambda^2} + 3\frac{w}{r+g+\Lambda^2}$$

For the renormalization of the couplings we find the following contributions:

	A	B	C	D	E	F	coupl.	propagators	diagrams	mult.
u (ϕ_α^4)	α	α	α	α	α	α	u^2	$(r+\Lambda^2)^{-2}$	1	9
	α	α	β	β	α	α	u^2	$(r+\Lambda^2)^{-2}$	$n-2$	1
	α	α	n	n	α	α	v	$(r+g+\Lambda^2)^{-2}$	1	1
u $(\phi_\alpha^2\phi_\beta^2)$	α	α	α	α	β	β	u^2	$(r+\Lambda^2)^{-2}$	1	3
	α	α	β	β	β	β	u^2	$(r+\Lambda^2)^{-2}$	1	3
	α	β	α	β	α	β	u^2	$(r+\Lambda^2)^{-2}$	4	1
	α	α	γ	γ	β	β	u^2	$(r+\Lambda^2)^{-2}$	$n-3$	1
	α	α	n	n	β	β	v^2	$(r+g\Lambda^2)^{-2}$	1	1
v	α	α	α	α	n	n	uv	$(r+\Lambda^2)^{-2}$	1	3
	α	α	β	β	n	n	uv	$(r+\Lambda^2)^{-2}$	$n-2$	1
	α	α	n	n	n	n	vw	$(r+g+\Lambda^2)^{-2}$	1	3
	α	n	α	n	α	n	v^2	$((r+\Lambda^2)(r+g+\Lambda^2))^{-1}$	4	1
w	n	n	α	α	n	n	v^2	$(r+\Lambda^2)^{-2}$	$n-1$	1
	n	n	n	n	n	n	w^2	$(r+g+\Lambda^2)^{-2}$	1	9

So we get:

$$\frac{\mathrm{d}u}{\mathrm{d}l} = \varepsilon u - (n+7)\frac{u^2}{(r+\Lambda^2)^2} - \frac{v^2}{(r+\Lambda^2)^2}$$
$$\frac{\mathrm{d}v}{\mathrm{d}l} = \varepsilon v - (n+1)\frac{uv}{(r+\Lambda^2)^2} - 3\frac{vw}{(r+g+\Lambda^2)^2} - 4\frac{v^2}{(r+g+\Lambda^2)(r+\Lambda^2)}$$
$$\frac{\mathrm{d}w}{\mathrm{d}l} = \varepsilon w - (n-1)\frac{v^2}{(r+\Lambda^2)^2} - 9\frac{w^2}{(r+g+\Lambda^2)^2}$$