Take Home Exam, PHY 6937-0002 Phase Transitions and Critical Phenomena

In the following problems, the more challenging questions are indicated by (*). I expects students in Condensed Matter Theory to complete these problems, since they need to become fluent in doing such technical calculations.

IMPORTANT: *In solving this Take-Home Final Exam, I expect each student to work INDEPENDENTLY, therefore WITHOUT consulting with, or seeking help from, other students in the class. Remember that you are bound by the Academic Honor Code to follow this requirement, which should be strictly respected in solving this Exam.*

Problem 1. Write a short discussion (one or two paragraphs) to answer
Tot.55 each of the following physical questions.

a) Explain the concept of the thermodynamic limit and its relevance to spon-
5 taneous symmetry breaking.

b) Which of the following systems can display **spontaneous** symmetry
20 breaking and a finite temperature second order phase transition **in presence of a weak but finite uniform magnetic field**: (i) A three dimensional Ising ferromagnet; (ii) A two dimensional Ising antiferromagnet; (iii) A three dimensional Heisenberg antiferromagnet; (iv) A two dimensional Heisenberg antiferromagnet. Explain the rationale for your answer.

c) Explain why the critical exponents assume the same values in any kind of
5 mean-field (or Landau) theory. Use the power-counting argument to substantiate
your answer.

d) How do the critical exponents depend on the precise form of the crystal
5 lattice? Explain.

e) Explain why there can be no finite temperature phase transition in any
5 one dimensional system with short-range interactions.

f) For which kinds of systems can there be no finite temperature phase tran-
5 sitions in two dimensions?

g) Under which conditions can one dimensional systems can display a finite
5 temperature phase transitions? Explain your answer on physical grounds.

h) Explain why quantum fluctuations are irrelevant at finite temperature
5 phase transitions.

Problem 2. Consider a lattice gas model describing the liquid-gas phase transition, as given by the Hamiltonian

Tot.45

$$H = -V \sum_{ij} n_i n_j - \mu \sum_i n_i .$$

Here, $n_i = 0, 1$ are the lattice site occupation numbers, and μ is the chemical potential. The interaction is chosen to be attractive ($V > 0$), so that the particles have a tendency to form a high-density (liquid) phase at low temperature. We have chosen to treat the problem in the grand canonical ensemble, where the total number of particles $N = \sum_i n_i$ is fixed only on the average.

a) Show that a linear transformation $n_i = \frac{1}{2}(S_i + 1)$, with $S_i = \pm 1$ reduces the problem to an equivalent ferromagnetic Ising model. Determine the corre-
5 sponding magnetic exchange J and the external field h in terms of the lattice gas parameters V and μ. Determine also how the average density $n = \langle n_i \rangle$ can be related to the magnetization $m = \langle S_i \rangle$ of the equivalent Ising model.

b) Using the mapping to the Ising model, determine the value of the chemical
5 potential μ corresponding to the average occupation at half-filling, i.e. $\langle n_i \rangle = 1/2$.

c) Use mean-field theory to construct the phase diagram of the model in the
5 $\mu - T$ plane, and determine the location of the liquid-gas critical point.

5 d) Solve the mean-field equation of state around the critical point and compute the density as a function of μ and T.

10 e) Use thermodynamic relations to determine the pressure P as a function of μ and T around the critical point.

10 f) Construct the phase diagram as a function of the pressure P and density n showing the liquid-gas coexistence region. On this phase diagram plot $P(n)$ for several different temperatures (i.e. plot isotherms), above and below T_c. Within the coexistence region the system will display **phase separation** so that the liquid and the gas phases occupy different parts of the system volume.

5 g) What is the upper critical dimension and the lower critical dimension for this model?

Problem 3*. Consider a one dimensional $O(N)$ model ($N > 1$) given by the Landau Action of the form

Theorist only

$$S = \frac{1}{2}\sum_{\alpha=1}^{N}\int dx\ \phi_\alpha(x)\left[r - \nabla^2\right]\phi_\alpha(x) + \frac{u}{4N}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\int dx\,\phi_\alpha^2(x)\,\phi_\beta^2(x).$$

Now imagine turning on an additional long-range interaction between the spins described by the Action

$$S_{lr} = -V\sum_{\alpha=1}^{N}\int dx \int dx'\,\frac{\phi_\alpha(x)\phi_\alpha(x')}{(x-x')^{1+\sigma}}.$$

10 a) Write down the full Action of this system in momentum space. Which term provides the leading contribution at low momenta? How does your conclusion depend on the value of the exponent σ? For which values of σ do you expect the behavior to be modified from that of the standard (shot-range) model?

20 b) Now examine the model in the large N limit, following the same strategy we used for the short-range model. To do this, decouple the ϕ^4-term using a Hubbard-Stratonovich transformation, and then solve the partition function using the saddle-point method (which is exact for $N \longrightarrow \infty$).

10 c) Determine the critical temperature T_c as a function of σ, and the behavior of the correlation length as T_c is approached. Is there something special for $\sigma = 1$? What happens when $\sigma > 1$?