

Problem 1

1 a) Usually thermodynamic limit refers to the case when the system size is infinity. The second order phase transitions are usually accompanied by a spontaneous symmetry breaking. As it turns out, a spontaneous symmetry breaking can not happen at finite systems. It only happens in thermodynamic limit. This can be easily justified based on the ergodicity breaking argument when system is prevented to occupy all its possible states. According to Boltzmann's ergodic hypothesis, at equilibrium system intends to go through all available states. The time it takes to reach each state is proportional to the probability of reaching each configuration and diverges exponentially with the system size. For finite size systems this time is relatively short, while in thermodynamic limit the system will be "stuck" for a long time in some subspace.

b) Spontaneous symmetry breaking can happen in (i) 3d Ising ferromagnet only at $h = 0$. In the presence of a weak external magnetic field, the symmetry of the system is already broken. So, there is no finite temperature second order transition in 3D Ising ferromagnet. At finite external field (ii) 2d antiferromagnet has second order phase transition because the external field does not effect staggered magnetisation. For the Heisenberg model there is no finite temperature phase transition in (iv) 2d Heisenberg antiferromagnet ($d_{lc} = 2$), but it can happen in (iii) 3d Heisenberg antiferromagnet.

c) Using the fact that close to the critical point, the order parameter is a small quantity, the free energy can be written as a Taylor expansion of the order parameter, leading to the given values of critical exponents which are independent of the numerical values of the coupling constant u, v, w , etc. The action is assumed to be an analytic function of the order parameter, consistent with the symmetries of the problem.

$$S = \int d^d x \{ \phi(x) (r - \nabla^2) \phi(x) \} + \frac{1}{4} u \int d^d x \phi^4(x) - \int d^d x j(x) \phi(x)$$

In any mean field theory such an action defines the corresponding values of the critical exponents. The particular form of the action depends on the symmetry and dimension of the problem. Power counting argument is important when one has to decide which term to drop in the action. For example, terms of the form $\int w_m \phi^m d^d x$ are irrelevant at $d(m) = \frac{2m}{m-2}$ and thus can be ignored.

d) The critical exponents do not depend on the precise form of the crystal lattice because all the information about the lattice is hidden in the coupling constants of corresponding mean field equations. At the same time, the values of the critical exponents do not depend on the coupling constants, but depend only on the form of the action. Thus, we may conclude that critical exponents depend on the symmetry of the order parameter, dimensionality, and the nature of the critical point; and are not effected by the values of coupling constants, type of the lattice or the form of the Hamiltonian.

e) In 1d system with short-range interaction, there is no finite temperature phase transition because moving the domain walls costs us no energy. The entropy $S \sim Ln(L)$ (L is the system size) always wins. Thus, putting domain walls will always lower the free energy no matter how low the temperature is. So, we can put the domain walls randomly positioned such that the ordered phase is destroyed at any finite temperature.

f) There can not be any finite temperature phase transition for continuous symmetry systems. It was shown that for a system with $O(N)$ symmetry in d -dimension, the critical temperature is $T_c \sim (d-2)/(N-2)$. Thus, for $N \geq 3$ the critical temperature $T_c = 0$ for $d = 2$.

g) 1d system can display a finite temperature phase transition if the interaction is long range. Since moving the domain wall costs us energy of order of L compared to the entropy $S \sim Ln(L)$. Then the free energy will be minimum at limited domain walls number which results in the ordered phase at finite temperature.

h) At finite T near the critical point, the correlation length diverges. Thus, the behavior of the system is determined by the regions whose size is larger than the length scale at which quantum effects are important. But even if one writes the quantum case of the Landau theory, the action looks the same as in the classical case but for $d+1$ -dimension. The imaginary time plays the role of the extra dimension. At finite temperature, however, there is a finite imaginary time cut-off. Close to T_c , spacial fluctuations are much larger than this cut-off, and we end up with the action which looks the same as the classical version of the problem.

Problem 2

Consider a lattice-gas model given by the Hamiltonian:

$$H = -V \sum_{\langle ij \rangle}^N n_i n_j - \mu \sum_i n_i, \quad (1)$$

where $n_i = 0, 1$ are the lattice site occupation numbers, and μ is a chemical potential.

a) Let

$$n_i = \frac{1}{2}(S_i + 1) \quad (2)$$

Then Hamiltonian in eq.1 is transformed as the following:

$$\begin{aligned} H_{LG} &= -\frac{V}{2} \sum_{i,j}^N n_i n_j - \mu \sum_i n_i \\ &= -\frac{V}{2} \sum_{i,j} \frac{1}{4} (S_i + 1)(S_j + 1) - \mu \sum_i \frac{1}{2} (S_i + 1) \\ &= -\frac{V}{2} \sum_{i,j} \frac{1}{4} S_i S_j - 2 \frac{V}{2} \sum_{i,j} \frac{1}{4} S_i - \frac{V}{2} \sum_{i,j} \frac{1}{4} 1 - \frac{\mu}{2} \sum_i S_i - \frac{\mu}{2} \sum_i 1 \\ &= -\frac{V}{4} \sum_{\langle i,j \rangle} S_i S_j - \left(\frac{Vz}{4} + \frac{\mu}{2} \right) \sum_i S_i - \frac{VzN}{8} - \frac{\mu N}{2}, \end{aligned} \quad (3)$$

here z is a number of the nearest neighbors.

This can be written as

$$H_{LG} = -J \sum_{\langle i,j \rangle} S_i S_j - h \sum_i S_i - E_0 \quad (4)$$

with

$$J = \frac{V}{4} \quad (5)$$

$$h = \frac{Vz}{4} + \frac{\mu}{2} \quad (6)$$

and constant

$$E_0 = \frac{VzN}{8} + \frac{\mu N}{2}. \quad (7)$$

We may also conclude that

$$H_{LG} = H_I - E_0 \quad (8)$$

and the grand partition function for the lattice-gas model can be written as

$$Z_{LG} = e^{\beta E_0} Z_I. \quad (9)$$

Z_I is the partition function of the Ising model.

From eq.2 we can determine the relationship between the average density $\langle n \rangle$ and magnetization $m = \langle S \rangle$. Simply averaging both sides of this equation we get

$$\langle n \rangle = \frac{1}{2}(\langle S \rangle + 1)$$

or

$$m = 2 \langle n \rangle - 1 \quad (10)$$

b) Taking into account that half-filling $\langle n \rangle = 1/2$ corresponds to $m = 0$ (from eq.10) at zero external field, we obtain from eq.6

$$0 = \frac{Vz}{4} + \frac{\mu}{2}$$

or

$$\mu = -\frac{Vz}{2}. \quad (11)$$

This can also be obtained from the direct calculation of the average density, i.e.

$$\langle n \rangle = \frac{1}{\beta} \frac{\partial \ln(Z_{LG})}{\partial \mu} = \frac{1}{\beta} \frac{\partial \ln(e^{\beta E_0} Z_I)}{\partial \mu}, \quad (12)$$

where the partition function of Ising model Z_I in the mean-field approach is

$$Z_I = Tr(e^{-\beta H_I}) = e^{-\frac{N\beta Jzm^2}{2}} (2\cosh(\beta h + \beta Jzm))^N. \quad (13)$$

Using this and expression for constant E_0 , we obtain:

$$\begin{aligned} \langle n \rangle &= \frac{1}{\beta} \left(\frac{\beta \frac{\partial h}{\partial \mu} 2 \sinh(\beta h + \beta Jzm)}{2 \cosh(\beta h + \beta Jzm)} + \beta \frac{\partial E_0}{\partial \mu} \right) \\ &= \frac{\tanh(\beta h + \beta Jzm)}{2} + \frac{1}{2} \end{aligned}$$

or

$$2 \langle n \rangle - 1 = \tanh \left(\beta \frac{Vz}{4} + \beta \frac{\mu}{2} + \frac{\beta Vz}{4} (2 \langle n \rangle - 1) \right) = \tanh \left(\frac{\beta Vz}{2} \langle n \rangle + \frac{\beta \mu}{2} \right) \quad (14)$$

Putting $\langle n \rangle = \frac{1}{2}$ in the last expression, we obtain the corresponding value of the chemical potential

$$\mu = -\frac{Vz}{2} \quad (15)$$

c) In the mean-field approach, the Weiss self-consistency condition reads

$$m = \tanh(\beta h + \beta z J m) \quad (16)$$

Substituting the found above expressions for m , h and J into eq.16, we find a corresponding condition for the lattice-gas model:

$$2 \langle n \rangle - 1 = \tanh\left(\frac{\beta Vz}{4} + \frac{\beta \mu}{2} + \frac{\beta Vz(2 \langle n \rangle - 1)}{4}\right)$$

or

$$2 \langle n \rangle - 1 = \tanh\left(\frac{\beta \mu}{2} + \frac{\beta Vz \langle n \rangle}{2}\right) \quad (17)$$

To determine the critical temperature, we fix the chemical potential $\mu = -\frac{V}{2}$, and obtain:

$$2 \langle n \rangle - 1 = \tanh\left(\frac{\beta Vz(2 \langle n \rangle - 1)}{4}\right) \quad (18)$$

This equation can be solved graphically:

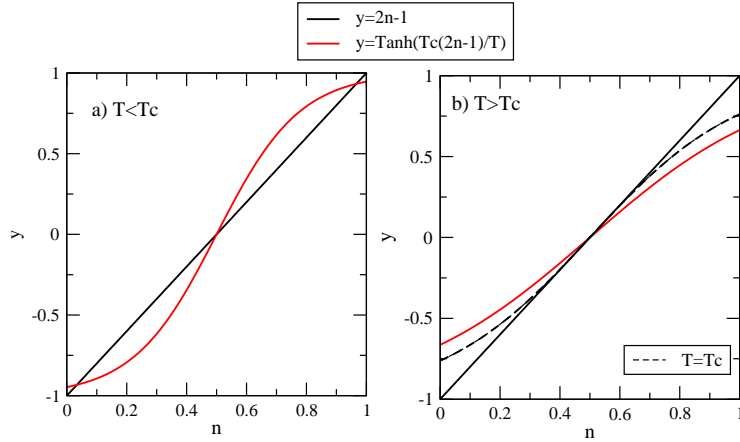


Figure 1: Graphical solution of “Weiss self-consistency condition” for the lattice-gas model at $\mu = -Vz/2$.

For $T < T_c$ there are two distinct solutions of eq.18, while for $T > T_c$ there is only one intersection at $\langle n \rangle = 0.5$.

At $T = T_c$ the only intersection is still at $\langle n \rangle = 0.5$ but now both curves have the same slope there. This allows us to determine T_c by calculating slopes of these two curves at $\langle n \rangle = 0.5$, i.e.

$$2 = \frac{zV}{2T_c}$$

or

$$T_c = \frac{Vz}{4} \quad (19)$$

We could have obtained this result with much less efforts, just by using the fact that for the Ising model $T_c = zJ$, accordingly, for the lattice gas model $T_c = zV/4$.

Plotting $\langle n \rangle$ as function of μ (Fig.2-a), we can see that at half-filling $\mu = -\frac{Vz}{2}$ there is a jump in the density $\langle n \rangle$ for $T < T_c$, just like in the Ising model at $h = 0$. Such a discontinuity in the order parameter is the feature of the first order transition.

The corresponding $\mu - T$ phase diagram is shown in Fig2-b. For a chemical potential $\mu < -Vz/2$ we obtain low density (negative magnetization) gas; and in the region where $\mu > -Vz/2$ we have a high density (positive magnetization) liquid.

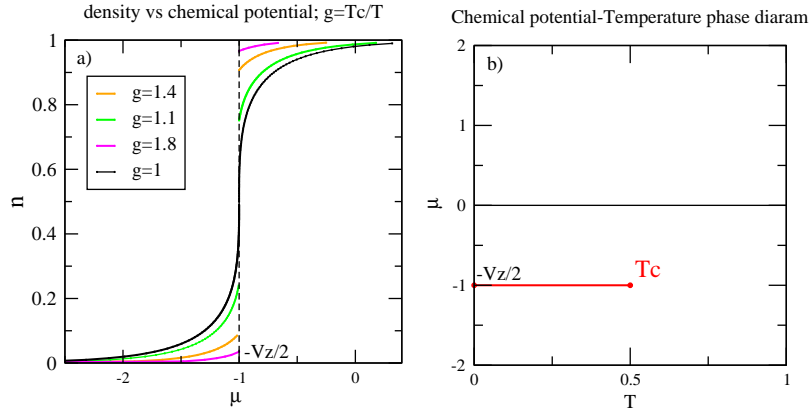


Figure 2: $\mu - T$ phase diagram

d) To find $\langle n \rangle$ as function of μ and T , we expand self-consistency condition (eq.17) around critical point T_c . Let $g = \frac{T_c}{T}$.

We first rewrite eq.17 in the form:

$$2 \langle n \rangle - 1 = \tanh \left(g(2 \langle n \rangle - 1) + \beta \left(\frac{Vz}{4} + \frac{\mu}{2} \right) \right), \quad (20)$$

where we used that $T_c = -Vz/4$.

We invert this expression to obtain the equation of states.

$$\tanh \left(\beta \left(\frac{Vz}{4} + \frac{\mu}{2} \right) \right) = \frac{(2 \langle n \rangle - 1) - \tanh(g(2 \langle n \rangle - 1))}{1 - (2 \langle n \rangle - 1) \tanh(g(2 \langle n \rangle - 1))}. \quad (21)$$

Near the critical point $2 < n > -1 \ll 1$ and $\frac{Vz}{4} + \frac{\mu}{2} \ll 1$ so we can expand eq.21 and obtain the equation of states in the form:

$$\beta\left(\frac{Vz}{4} + \frac{\mu}{2}\right) \approx (2 < n > -1)(1 - g) + (2 < n > -1)^3\left(g - g^2 + \frac{1}{3}g^3\right) \quad (22)$$

Now we consider:

1) $\frac{\mu}{2} = \frac{Vz}{4}$ (Ising case: $h = 0$) and $T \rightarrow T_c$

Then eq.22 implies:

$$0 \approx T(2 < n > -1)\left(1 - \frac{T_c}{T}\right) + T(2 < n > -1)^3\left(g - g^2 + \frac{1}{3}g^3\right)$$

or

$$2 < n > -1 \approx \left(\frac{3(T_c - T)}{T_c}\right)^{1/2} \quad (23)$$

2) $\frac{\mu}{2} \neq \frac{Vz}{4}$ (Ising case: $h \neq 0$) but $T = T_c$

Putting $g = 1$ in eq. 22, we get

$$\frac{Vz}{4} + \frac{\mu}{2} \approx (2 < n > -1)^3$$

or

$$< n > \approx \frac{1}{2} \left(\frac{Vz}{4} + \frac{\mu}{2}\right)^{1/3} + \frac{1}{2} \quad (24)$$

e) Pressure can be calculated from

$$p = -\frac{F}{V} = \frac{Ln(Z_{LG})}{\beta V}, \quad (25)$$

where F is a free energy of the lattice-gas model, $V = V_0N$ is the volume of the system, V_0 is the volume of a unite cell, and N is the total number of the lattice sites.

Taking into account expression for Z_{LG} (eq. 9) and Z_I (eq. 13), we obtain

$$\begin{aligned} p &= \frac{E_0}{N} + \frac{Ln(Z_I)}{\beta V} \\ &= \frac{Vz}{8} + \frac{\mu}{2} - \frac{N\beta Jzm^2}{\beta N} + \frac{N(Ln2 + Lncosh(\beta h + \beta Jzm))}{\beta N} \end{aligned} \quad (26)$$

or

$$p = \frac{Vz}{8} + \frac{\mu}{2} - \frac{Vz}{8}(2 < n > -1)^2 + \frac{\left(Ln2 + Lncosh\left(\frac{\beta Vz}{4} + \frac{\beta \mu}{2} + \frac{\beta Vz}{4}(2 < n > -1)\right)\right)}{\beta}$$

Finally,

$$p = -\frac{Vz}{2} \langle n \rangle^2 + \frac{Vz}{2} \langle n \rangle + \frac{\mu}{2} + \frac{Ln2}{\beta} + \frac{Lncosh\left(\frac{\beta Vz}{2} \langle n \rangle + \frac{\beta \mu}{2}\right)}{\beta} \quad (27)$$

f) Below we present the phase diagram as function of the pressure p and average density $\langle n \rangle$ (Fig.3).

To obtain this diagram, one can eliminate a chemical potential μ from eq. 27 using self-consistency condition (eq. 17) for different values of $\langle n \rangle$. Then plot the pressure p as function of the density $\langle n \rangle$ at corresponding values of μ at some constant temperature T .

In Fig 3 we plotted p as function of $\langle n \rangle$ for temperatures above and below T_c . The dashed line indicates the values of p and $\langle n \rangle$ at half-filling $\mu = -Vz/2$ ($h = 0$). For $T < T_c$ and $\mu = -Vz/2$ there are two solutions of eq.17 with the same pressure (free energy), while above T_c system has only one solution. Thus, below T_c at $\mu = -Vz/2$ system exhibits phase separation, i.e. there is a coexistence of the liquid and the gas in the system.

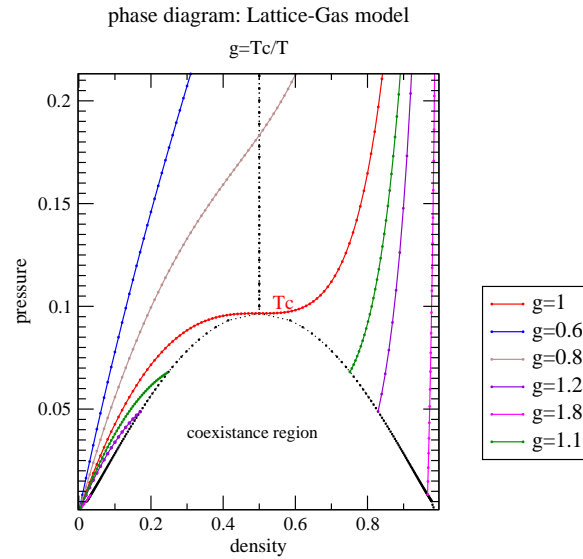


Figure 3: p-n phase diagram. $V = 2$, $g = T_c/T$.

g) In this problem, using a linear transformation we have mapped the lattice gas model into the magnetic Ising model.

Consequently, the lower and upper critical dimensions should be the same as in the Ising model case. Thus,

$$d_{lc} = 1 \quad (28)$$

and

$$d_{uc} = 4 \quad (29)$$

Problem 3

3. a) We use the change variables

$$\begin{aligned}\epsilon &= (x + x')/2 \\ \zeta &= (x - x') \\ \text{Jacobian} \left| \frac{\partial(x, x')}{\partial(\epsilon, \zeta)} \right| &= 1\end{aligned}$$

Then Fourier transform of S_{lr} is

$$S_{lr} = -V \sum_{\alpha=1}^N \int d\epsilon d\delta dk_1 dk_2 \left[\tilde{\phi}_\alpha(k_1) \tilde{\phi}_\alpha(k_2) \right] \frac{e^{i(k_1+k_2)\epsilon} e^{i(k_1-k_2)\zeta/2}}{(2\pi)^2 \zeta^{1+\sigma}} \quad (30)$$

Integrating over ϵ gives us a delta Dirac function $\delta(k_1 + k_2)$, and using $y = \zeta k$ we obtain:

$$\begin{aligned}S_{lr} &= -V \sum_{\alpha=1}^N \int d\zeta dk_1 dk_2 \left[\tilde{\phi}_\alpha(k_1) \tilde{\phi}_\alpha(k_2) \right] \frac{e^{i(k_1-k_2)\delta/2}}{(2\pi)\zeta^{1+\sigma}} \delta(k_1 + k_2) \\ &= -V \sum_{\alpha=1}^N \int dy \frac{e^{iy}}{y^{1+\sigma}} \int dk_1 \frac{\tilde{\phi}_\alpha(k) k^\sigma \tilde{\phi}_\alpha(-k)}{2\pi}\end{aligned} \quad (31)$$

As a result, we get the square term (eq. 32) adding to the ordinary short range ϕ^4 model

$$\begin{aligned}S &= \frac{1}{2} \sum_{\alpha=1}^N \int dk \frac{\tilde{\phi}_\alpha(k)(r + k^2 + wk^\sigma) \tilde{\phi}_\alpha(-k)}{2\pi} \\ &+ \frac{u}{4N(2\pi)^4} \int dk_1 dk_2 dk_3 \tilde{\phi}_\alpha(k_1) \tilde{\phi}_\alpha(k_2) \tilde{\phi}_\beta(k_3) \tilde{\phi}(-k_1 - k_2 - k_3)\end{aligned} \quad (32)$$

So we have different form of the action S of ordinary ϕ^4 - theory in small k expansion when $0 < \sigma < 2$. In this case, we can ignore k^2 term in the quadratic part. If $\sigma \geq 2$, we have the same result as the ordinary ϕ^4 - theory (discussed in the lecturer notes)

b) We are interesting in the range $0 < \sigma < 2$. Using Hubbard-Stratonovich transformation in real space we get:

$$\exp\left(\frac{1}{4N} \int dx \phi^4\right) \rightarrow \int d\psi \exp\left(\frac{1}{2}u \int dx \psi \phi^2 - \frac{N}{4}u \int dx \psi^2\right)$$

$$Z = \int D\psi D\tilde{\phi} D\tilde{\phi}^* \exp\left\{ \frac{1}{8\pi} \sum_{\alpha=1}^N \int dk |\tilde{\phi}|^2 (r + u\zeta + w|k|^\sigma) - \frac{1}{4}Nu \int dx \psi^2(x) \right\} \quad (33)$$

Integrating all states except $\alpha = 1$ and rewriting the equation for the action (eq.33) with rescaling $\phi_{1'} \equiv \phi_1/\sqrt{N}$: In the large N limit we can use saddle point method:

$$S = \frac{1}{4\pi}N \int dk \phi_{1'}(k)(r + u\psi(k) - |k|^\sigma)\phi_{1'}(k) - \frac{1}{8\pi}Nu \int dk \psi^2(k) - (N-1) \text{Log} \int D\tilde{\phi}_\alpha \exp \left\{ \frac{1}{4\pi} \int dk \tilde{\phi}_\alpha(k)(r + u\psi + w|k|^\sigma)\tilde{\phi}_\alpha(k) \right\} \quad (34)$$

$$\begin{aligned} \frac{\delta S}{\delta \phi_{1'}}|_{Sp} &= 0 \\ \frac{\delta S}{\delta \psi}|_{Sp} &= 0 \end{aligned}$$

from equation (5) we get :

$$\begin{aligned} \int dk \phi_{1'}(r + u\psi_o(k)) &= 0 \\ r + u\psi_o(k) &= 0 \end{aligned} \quad (35)$$

and

$$\int \frac{dk}{2\pi} \left\{ \phi_o^2 u - \frac{u}{2} \cdot 2\psi_o + \frac{\int D\phi_\alpha \phi u \phi_\alpha \exp(-\frac{1}{4\pi} \int dk \phi_\alpha(r + u\psi_o + w|k|^\sigma)\phi_\alpha)}{2 \int D\phi_\alpha \exp(-\frac{1}{4\pi} \int dk \phi_\alpha(r + u\psi_o + w|k|^\sigma)\phi_\alpha)} \right\} = 0$$

$$u\phi_o^2 - u\psi_o - \int \frac{dk}{2\pi} \frac{u}{r + u\psi_o + w|k|^\sigma} = 0 \quad (36)$$

So we will get :

$$\phi_o = \sqrt{\frac{-r}{u} - \int \frac{dk}{2\pi w|k|^\sigma}} \quad (37)$$

At the critical temperature $\phi_o = 0$,

$$r_c = \frac{-u}{w} \int_0^\Lambda \frac{dk}{2\pi|k|^\sigma} \quad (38)$$

converge if $0 < \sigma < 1$

$$r_c = -\frac{1}{2\pi} \frac{u}{w} \frac{\Lambda^{1-\sigma}}{1-\sigma} \quad (39)$$

Subceptibility can be find by setting $\phi_o = 0$ in the saddle -point condition and $\bar{r} = r + u\psi_o$, we can get the equation:

$$\begin{aligned}
\chi^{-1} = \bar{r} &= r + \int_0^\Lambda \frac{u dk}{2\pi(\bar{r} + w|k|^\sigma)} r \\
\bar{r} &\approx r + \frac{1}{2\pi} \int_{(\bar{r}/w)^{1/\sigma}}^\Lambda \frac{u k^{-\sigma} dk}{w} \\
&= r + \frac{u}{2\pi w} \left\{ \frac{\Lambda^{1-\sigma} - (\bar{r}/w)^{1/\sigma-1}}{1-\sigma} \right\}
\end{aligned} \tag{40}$$

finally, we can rewrite this equation as:

$$r + \delta r = \bar{r} + \frac{u\bar{r}^{1/\sigma-1}}{2\pi w^{1/\sigma-1}(1-\sigma)} \tag{41}$$

We can see that the linear term dominates when $\sigma < 1/2$ and $\gamma = 1$ (mean field theory). If $\sigma > 1/2$ the second term dominates and we have $\gamma = \sigma/(1-\sigma)$.

c) The correlation length can be found by determining non local susceptibility at $r \approx 0$

$$\begin{aligned}
\chi(x) &= \frac{1}{2\pi} \int_0^\infty \frac{dk e^{ikx}}{k^\sigma} \\
&= \frac{1}{\pi} \Gamma(1-\sigma) \sin\left(\frac{\pi\sigma}{2}\right) \frac{1}{x^{1-\sigma}} \\
\chi(x) &\propto \frac{1}{x^{1-\sigma}}
\end{aligned} \tag{42}$$

In our case $d = 1$ we also have the definition of anomalous dimension $\chi(x) \propto x^{1-\eta}$, therefore $\eta = 2 - \sigma$.

Use Kadanoff relation to find other exponents:

$$\begin{aligned}
\gamma &= \sigma\nu \\
\text{if } \sigma < 1/2 \text{ then } \nu &= 1/\sigma,
\end{aligned} \tag{43}$$

$$\text{if } \sigma > 1/2 \text{ then } \nu = 1/(1-\sigma) \tag{44}$$

We conclude that at $\sigma > 1$ our result for γ and ν becomes infinite and there is no phase transition. At $\sigma < 1$, exponents γ, ν are finite and phase transition exist at finite temperature. Special case at $\sigma = 1$, γ, ν are also logarithmically divergent or we can say at $\sigma = 1$ (marginal case), there is no phase transition at finite temperature.