Renormalization of the Gaussian Model

The renormalization group analysis of the ϕ^4 theory can be performed exactly above the upper critical dimension d_{uc}, where the Gaussian version of the theory is sufficient to describe the critical behavior.

Gaussian Model: the "free" theory

We have seen how the Ising model partition function can be rewritten as a functional integral

$$Z[h] = \int D\phi \; e^{-S[\phi]},$$

with an Action of the form

$$S[\phi] = \frac{1}{2}\int d\mathbf{x}\,\phi(\mathbf{x})\,[r - \nabla^2]\,\phi(\mathbf{x}) + \frac{u}{4}\int d\mathbf{x}\;\phi^4(\mathbf{x}) - \int d\mathbf{x}\,j(\mathbf{x})\phi(\mathbf{x}).$$

As we mentioned, such a partition function is identical to the "generating functional" $Z[j]$ of an Euclidian (imaginary time) ϕ^4 field theory. The expectation value (i.e. thermal average) of any physical quantity can be obtained by taking appropriate (functional) derivatives with respect to the external fields $j(\mathbf{x})$. For example, the order parameter can be written as

$$\langle\phi\rangle = \frac{\delta}{\delta\, j(\mathbf{x})}\ln Z[j].$$

Similarly, the correlation function

$$G(\mathbf{x}-\mathbf{x}') = \langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle_c = \langle\phi(\mathbf{x})\phi(\mathbf{x}')\rangle - \langle\phi(\mathbf{x})\rangle\,\langle\phi(\mathbf{x}')\rangle = \frac{\delta}{\delta\, j(\mathbf{x})\delta\, j(\mathbf{x}')}\ln Z[j].$$

[Here we have switched to the standard field-theoretic notation, where the external field is denoted by $j(\mathbf{x})$, and the correlation function by $G(\mathbf{x}-\mathbf{x}')$. The expression $\langle ...\rangle_c$ indicates a "connected" correlation function, i.e. a cumulant as oppose to a moment.]

Interpreted as a field theory, this model describes a set of interacting relativistic bosons with mass $m = r^{1/2}$, and an interaction amplitude u. In general we do not know how to solve such a field theory, but the situation is much simpler in the "noninteracting" case $u = 0$, which corresponds to the Gaussian model given by the quadratic part of the Action

$$S_o[\phi] = \frac{1}{2}\int d\mathbf{x}\,\phi(\mathbf{x})\,[r - \nabla^2]\,\phi(\mathbf{x}) - \int d\mathbf{x}\,j(\mathbf{x})\phi(\mathbf{x}).$$

This corresponds to free propagating bosons, and the $Z[j]$ is easy to explicitly compute since it is simply a Gaussian integral. Up to an (irrelevant) numerical prefactor we find

$$Z_o[j] = \exp\left\{\frac{1}{2}\int d\mathbf{x}d\mathbf{x}' j(\mathbf{x})G_o(\mathbf{x}-\mathbf{x}')j(\mathbf{x})\right\},$$

where $G_o(\mathbf{x}-\mathbf{x}')$ is simply the "bare" Green's function

$$G_o(\mathbf{k}) = [r+k^2]^{-1}.$$

Note that within such a free theory, and correlation functions of higher order "factor out" in products of the bare Green's functions, and we arrive at the World's easiest derivation of Wick's theorem! For example, the vertex function

$$\Gamma_4(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) = \langle\phi(\mathbf{x}_1)\phi(\mathbf{x}_2)\phi(\mathbf{x}_3)\phi(\mathbf{x}_4)\rangle_c = \frac{\delta^4}{\delta j(\mathbf{x}_1)\delta j(\mathbf{x}_2)\delta j(\mathbf{x}_3)\delta j(\mathbf{x}_4)}\ln Z[j]$$
$$= G_o(\mathbf{x}_1-\mathbf{x}_2)G_o(\mathbf{x}_3-\mathbf{x}_4) + G_o(\mathbf{x}_1-\mathbf{x}_3)G_o(\mathbf{x}_2-\mathbf{x}_4) + G_o(\mathbf{x}_1-\mathbf{x}_4)G_o(\mathbf{x}_2-\mathbf{x}_3).$$

We will find these expressions useful later, when we discuss the effects of the "interaction" terms such as $\frac{u}{4}\int d\mathbf{x}\ \phi^4(\mathbf{x})$.

We are now in a position to immediately compute the physical quantities within such a free theory. We have already calculate the long distance form of the free propagator

$$G_o(R) \sim \exp\{-R/\xi\},$$

identifying the correlation length $\xi = r^{1/2}$. This the correlation length exponent $\nu = 1/2$, just as in Landau theory! Similarly, the bulk susceptibility

$$\chi = G_o(\mathbf{k}=\mathbf{0}) = r^{-1},$$

and the susceptibility exponent $\gamma = 1$, again as in Landau theory. is this a coincidence? Not really! We have already seen that Landau theory emerges as a saddle-point solution of the ϕ^4 field theory. The correlations we examine within the Gaussian model correspond to the small Gaussian fluctuations around this saddle-point solution (the expressions written above correspond to the disordered phase where $\langle\phi\rangle = 0$. However, just as we did when we examined the Landau theory, a similar analysis can be carried out also within the ordered phase, by expanding around the nonzero value of $\langle\phi\rangle = \phi_o$, and again retaining only the terms quadratic in the fluctuating field to obtain another Gaussian model).

Renormalization of the Gaussian model

To obtain a deeper understanding of the limitations of such a Gaussian approximation, we now proceed to perform a RG analysis. To do this, we follow the philosophy of Kandanoff, which suggests "coarse-graining", trying to eliminate the short-wavelength fluctuations of the order parameter field. In the continuum model we now examine, this can be made precise by separating the long and the short wavelength components of $\phi(\mathbf{x})$, as follows

$$\phi(\mathbf{x}) = \phi_{long}(\mathbf{x}) + \phi_{short}(\mathbf{x}),$$

where $\phi_l(\mathbf{x})$ corresponds to the long-wavelength components (wavevector $0 < k < \Lambda/b$; here the length rescaling factor $b > 1$, as before, while $\Lambda \sim a^{-1}$ is the "ultraviolet cutoff set by the lattice spacing)

$$\phi_{long}(\mathbf{x}) = \int_0^{\Lambda/b} \frac{d\mathbf{k}}{(2\pi)^d}\phi(\mathbf{k}).$$

The short-wavelength part is

$$\phi_{short}(\mathbf{x}) = \int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d}\phi(\mathbf{k}).$$

Using such a decomposition, the Gaussian Action can be written as

$$S_o[\phi] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d\mathbf{k}}{(2\pi)^d}(\mathbf{k})\,\phi_{long}[r{+}k^2]\,\phi_{long}(\mathbf{k}) + \frac{1}{2}\int_{\Lambda/b}^{\Lambda} \frac{d\mathbf{k}}{(2\pi)^d}\,\phi_{short}(\mathbf{k})\,[r{+}k^2]\,\phi_{short}(\mathbf{k}) - \int d\mathbf{x}\, j(\mathbf{x})\phi(\mathbf{x}).$$

[Note that no cross-terms emerge, since the Action is diagonal in momentum space.] We can now immediately integrate over $\psi(\mathbf{x})$ to write

$$Z = Z_{short}\, Z_{long},$$

where

$$Z_{short} = \exp\left\{ \frac{1}{2}\int_{\Lambda/b}^{\Lambda} d\mathbf{k}\; j(\mathbf{k}) G_o(\mathbf{k})\, j(\mathbf{k}) \right\}$$

This part of the partition function is not very important for the critical behavior, since it describes only fluctuations with short wavelength. Since only $G_o(0)$ diverges at a critical point, Z_{short} only contributes to a smooth "background" component of the free energy, which is unimportant for the critical behavior, and which we therefore ignore from now on. We concentrate on the properties of Z_{long}, which corresponds to the long-wavelength action of the form

$$S_o[\phi] = \frac{1}{2}\int_0^{\Lambda/b} \frac{d\mathbf{k}}{(2\pi)^d}\phi_{long}(\mathbf{k})\,[r{+}k^2]\,\phi_{long}(\mathbf{k}) - j\int_0^{\Lambda/b} \frac{d\mathbf{k}}{(2\pi)^d}\,\delta^d(\mathbf{k})\phi_{long}(\mathbf{k}).$$

Here, we have used the fact that we have a uniform external field, so $j(\mathbf{k}) = j\delta^d(\mathbf{k})$. As we can see, this Action has precisely the same form as the original one, except for the fact that the ultraviolet cutoff now is Λ/b. How can we restore the action to the original form? Easy! We simply *rescale* all the lengths as follows: $\mathbf{k} \longrightarrow \mathbf{k}/b$, $d\mathbf{k} \longrightarrow b^{-d}d\mathbf{k}$ in momentum space, i.e. $\mathbf{x} \longrightarrow b\mathbf{x}$, $d\mathbf{x} \longrightarrow b^d d\mathbf{x}$ in real space. In order for the coefficient of k^2 to remain parameter-free, we also need to introduce a "wavefunction renormalization"

$$\phi_{long}(\mathbf{k}) \longrightarrow b^{1+d/2}\phi(\mathbf{k}),$$

in momentum space, or

$$\phi_{long}(\mathbf{x}) \longrightarrow b^{1-d/2}\phi(\mathbf{x}),$$

in real space. To renormalize the field term we follow a similar procedure, which is more convenient to carry out in real space (to avoid dealing with $\delta^d(\mathbf{k})$).

The action now assumes the form identical as before, except for the renormalized values for the coupling constants

$$r(b) = b^2 r,$$
$$j(b) = b^{1+d/2} j.$$

Following Kadanoff's prescription, we immediately read-off

$$\lambda_r = 2; \quad \lambda_j = 1 + d/2,$$

consistent, for example, with $\nu = \lambda_r^{-1} = 1/2$.