Broken symmetry and Goldstone modes

In presence of spontaneously broken continuous symmetry new types of excitations appear, which cost very little energy at long wavelength. Such "Goldstone modes" are, in fact, nothing but sound modes propagating through the solid. They lead to long-range correlations within the ordered phase, and give rise to large fluctuation effects in low dimensions. As a result any finite temperature ordering is suppressed in dimension $d \leq 2$, for systems with continuous symmetry. We establish this result first by examining the low energy fluctuations around the ground state. Similar conclusions are then shown to emerge from the Landau-Ginzburg formulation, which we solve exactly in the large-N limit.

So far we have concentrated much attention on the simplest example of critical phenomena - the Ising model. Here, the order parameter displays discrete symmetry, as it can point only in two directions ("up or down"). In this case, any excitations above the ground state cost a finite energy, since a domain wall has to be created to reverse the spin in a given region. In other words, we have a finite *gap* for elementary excitations, which for the ground state of the Ising model is of the order $E_g \sim J$ (energy to flip a single spin). As a result, excitations are created as an activated process, i.e. their density is of the order $\exp\{-\ E_g/T\}$.

In many ordered phases, however, there exists an order parameter characterized by continuous symmetry. A prototype is the Heisenberg (isotropic) ferromagnet, where the magnetization $\mathbf{m} = \langle \sigma_i \rangle$ is a three component ($N = 3$) vector that can point in any direction. In the ground state all the spins are aligned, and one may examine the cost of different low energy excitations. The energy of any configuration is described by the Heisenberg Hamiltonian

$$H = -J \sum_{ij} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i = J \sum_{ij} \cos(\theta_{ij}).$$

Here, $\boldsymbol{\sigma}_i = (\sigma_x, \sigma_y \sigma_z)$ is a unit-length ($|\boldsymbol{\sigma}_i| = 1$) vector that can point in any direction, and $\theta_{ij} = \arccos(\sigma_i \cdot \sigma_i)$ is their relative angle.

Let us assume that in the ground state all the spins point in the x-direction, so that $\mathbf{m} = (1, 0, 0)$. How much energy do we have to pay to rotate some of the spins by an infinitesimal angle θ? Well... to rotate only one of them will cost energy $E_g \sim J \cos(\theta)$.

Can't we do better then that? In fact...we can? We can rotate ALL the spins by the same infinitesimal angle θ, and this will cost us...NOTHING!!! This simple fact is a consequence of the invariance of the Hamiltonian with respect to a *global* rotation of all the spins by an arbitrary angle. We can immediately guess that a deformation that will cost us not zero, but in fact very little energy is the one that is an *almost* uniform rotation, i.e. the one corresponding to a long-wavelength *spin wave*.

Nonlinear σ-model

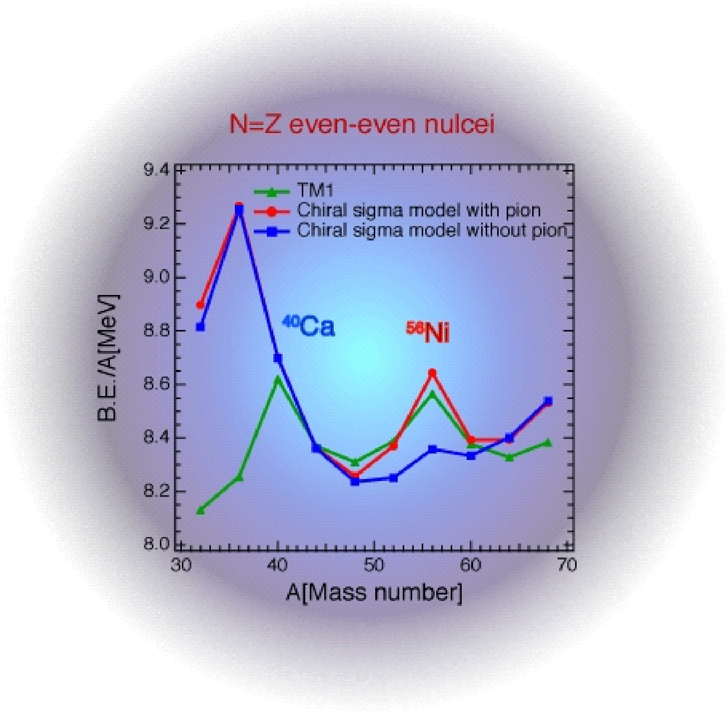
We can be more precise. Let us assume that the neighboring spins are almost aligned, so that the local spin direction changes very slowly from site to site. In this case it makes sense to introduce a continuous notation

$$H = -J \sum_{ii} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_i = J \sum_{ij} (\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_i)^2 + const. \approx \frac{1}{2} \kappa \int d\mathbf{x} \, (\nabla \boldsymbol{\sigma}(\mathbf{x}))^2 .$$

The parameter $k \sim a^{2-d} J$ is called the spin wave "stiffness". The energy cost of introducing a long-wavelength deformation of the spin field with wavevector q is $E(q) \sim \kappa q^2$. We can, therefore view the Heisenberg magnet as a deformable elastic medium, where the deformation we described are nothing but the *transverse acoustic waves* propagating through the spin system. As we will see, these wave excitations dominate the behavior in low dimensions. To examine this regime, we can write the partition function in the form

$$Z[j] = \int D\boldsymbol{\sigma}(\mathbf{x})\; \delta[\boldsymbol{\sigma}^2(\mathbf{x}) - \mathbf{1}]\; \exp\left\{-\frac{1}{2g}\int d\mathbf{x}\; (\nabla\, \boldsymbol{\sigma}(\mathbf{x}))^2 + \int d\mathbf{x}\; \mathbf{j}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})\right\},$$

where the coupling constant $g = T/\kappa$. This model has a fancy name, it's often called the **"nonlinear σ-model".** Historically, this model was first discussed in the context of meson theory in nuclear physics.



The situation we describe is, in fact, mush more general then the specific example we considered. A similar situation in found in ordinary elastic solids, where we consider spatial deformations of a crystalline lattice away from perfect periodicity. Again, moving all the atoms by the same amount cost us no energy, so the low energy excitations correspond to the long wavelength sound modes. Other examples include superfluids, superconductors...the list is long. In every instance there exists a global symmetry operation that leaves the energy of the system invariant. In each case, then, the low energy excitations assume a form of long-wavelength "hydrodynamic" modes. In field theory these are called "Goldstone bosons", since Goldstone proved a theorem showing that such excitations generally emerge as a consequence of broken global symmetry.

Lindemann criterion

To get a first hint how significant are these spin wave excitations in low dimensions, we present a heuristic argument that estimates the size of fluctuations around the ground state. Let us consider a low temperature situation, where the fluctuations are small, and the spin vector $\boldsymbol{\sigma}$ deviates little from its average value $\mathbf{m}$, which we chose to lie in direction $\alpha = 1$ (i.e. $\mathbf{m} = (1, 0, 0)$). Note that, due to the constraint $\boldsymbol{\sigma}^2 = \mathbf{1}$ all the components of $\boldsymbol{\sigma}$ are not independent. Let us denote the components as follows $\boldsymbol{\sigma} = (\sigma, \pi_1, \pi_2)$, and eliminate σ from the constraint

$$\sigma = \left(1 - \sum_{\alpha=1,2} \pi_\alpha^2\right)^{1/2} .$$

Now, if the fluctuations are small, then we can retain only the quadratic term

$$\sigma \approx 1 - \frac{1}{2} \sum_{\alpha=1,2} \pi_\alpha^2,$$

and our σ-model action take the form (at $j = 0$)

$$S \approx \frac{1}{2g} \int d\mathbf{x} \, (\nabla \pi(\mathbf{x}))^2 = \frac{1}{2} \sum_{\alpha=1,2} \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{\mathbf{q}^2}{g} \pi_\alpha^2(\mathbf{q}).$$

Note that the fields $\pi_\alpha(\mathbf{x})$ are not subject to any constraint. The Hamiltonian now looks like that of $n-1$ transverse sound modes. Since any wave is nothing but a collection of classical harmonic oscillators, the partition function is nothing but a familiar Gaussian integral, and we find

$$\left\langle \, (\pi_\alpha(\mathbf{x}))^2 \right\rangle \approx g \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{\mathbf{q}^2}.$$

We take a moment to examine this important result. Let us consider a system in a box of size L, so that the momenta acquire an infrared cutoff of order $2\pi/L$. Then

$$\int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{\mathbf{q}^2} = \frac{S_d}{(2\pi)^d} \int_{2\pi/L}^{2\pi/a} dq \, q^{d-3} = \frac{S_d}{(2\pi)^2} \frac{1}{d-2} [a^{2-d} - L^{2-d}].$$

The integral converges for $d > 2$, and can let $L \longrightarrow \infty$, to get

$$\left\langle \, (\pi_\alpha)^2 \right\rangle \approx \frac{T}{\kappa} \frac{S_d}{(2\pi)^2} \frac{1}{d-2} a^{2-d}.$$

We are now in a position to estimate the critical temperature. Lindemann developed a criterion for melting of crystals using the following argument. He suggested that melting will occur around the temperature where the fluctuation size (of molecules in a solid) becomes comparable to the interparticle distance. This heuristic criterion can be used for any elastic solid, and proves to work remarkably well for many materials.

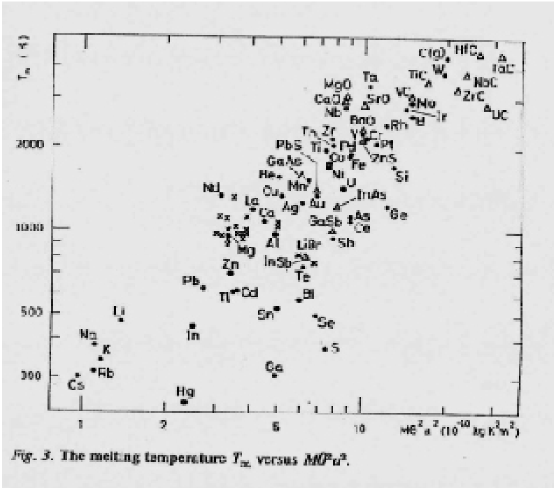


FIG. 1: The measured melting temperature versus the melting temperature estimated using the Lindemann rule, from: G. Grimvall and S. Sjodin, Physica Scripta, **10**, 340, (1974).

In our case, we should require that $\left\langle\, (\pi_\alpha)^2 \right\rangle \sim O(1)$, i.e. that the spin fluctuations become comparable to the ordered moment. Using this criterion, we estimate

$$T_c/J \sim (d-2)\,.$$

Note that the critical temperature is predicted to *decrease* as dimensionality is reduced, and to VANISH at $d \leq 2$!!! We conclude that the spin-wave fluctuations destroy order at and below the lower critical dimension which, for the models with continuous symmetry, is $d_{lc} = 2$. We can also see this by evaluating the integral directly in $d = 2$ giving

$$\left\langle\, (\pi_\alpha)^2 \right\rangle = \frac{g}{2\pi} \ln(L/a).$$

The fluctuations logarithmically BLOW UP with the system size in $d = 2$! Out theory, which was based on ASSUMING that fluctuations are small then proves incorrect, and the ordered phase is destroyed. Note that the opposite is true in $d > 2$, since our result predicts that the fluctuations can be made as small as we want, provided that the temperature T is low enough. The ordered phase is stable in that case, and will be destroyed only at a finite transition temperature T_c.

Goldstone's Theorem

We have presented simple arguments showing how infinitesimal excitations around the ground state have a character of propagating sound modes in a spin system with continuous symmetry. However, our argument was presented for a specific model, and furthermore it remained unclear if this feature is relevant at finite temperatures (still within the ordered phase), where the fluctuations cannot be considered as small. In the following we demonstrate what is known as the Goldstone's Theorem, which shows how the emergence of long-range correlations arises as a generic feature of broken continuous symmetry.

We concentrate on the form of the spin-spin correlation function

$$G_{\alpha\alpha}(\mathbf{x}) = \langle \phi_\alpha(\mathbf{x})\phi_\alpha(\mathbf{0})\rangle - \langle \phi_\alpha(\mathbf{x})\rangle \langle \phi\alpha(\mathbf{0})\rangle .$$

Technically, it can be evaluated by taking functional derivatives of the functional

$$F[j] = \ln Z[j],$$

as

$$G_{\alpha\alpha}(\mathbf{x}-\mathbf{y}) = \frac{\delta^2}{\delta j_\alpha(\mathbf{x})\delta j_\alpha(\mathbf{y})} F[j].$$

[Note that for an isotropic system correlators $G_{\alpha\beta}(\mathbf{x})$ with $\alpha \neq \beta$ vanish by symmetry].

Consider a system with continuous symmetry ($N > 1$), in the ordered phase, and imagine applying a small external field $j_a(\mathbf{x})$. Since the system is assumed to be isotropic, the partition function $Z[j]$, which is a scalar quantity, must be independent of the *direction* of the order parameter. It will, therefore remain unchanged if we perform an infinitesimal *rotation* of the field direction by an angle $\delta\theta$ in the (β, γ) plane. Only the components $j_\beta(\mathbf{x})$ and $j_\gamma(\mathbf{x})$ are affected

$$j'_\beta(\mathbf{x}) = j_\beta(\mathbf{x}) - \delta\theta j_\gamma(\mathbf{x}),$$
$$j'_\gamma(\mathbf{x}) = j_\gamma(\mathbf{x}) + \delta\theta j_b(\mathbf{x}).$$

The variation of $F[j]$ leads to

$$0 = \int d\mathbf{x} \left[\frac{\delta F}{\delta j_\beta(\mathbf{x})} j_\gamma(\mathbf{x}) - \frac{\delta F}{\delta j_\gamma(\mathbf{x})} j_\beta(\mathbf{x}) \right].$$

Tanking another variation with respect to $j_\gamma(\mathbf{y})$ gives

$$\begin{aligned} 0 &= \int d\mathbf{x} \left[\frac{\delta^2 F}{\delta j_\beta(\mathbf{x})\delta j_\gamma(\mathbf{y})} j_\gamma(\mathbf{x}) + \frac{\delta F}{\delta j_b(\mathbf{x})}\delta(\mathbf{x}-\mathbf{y}) - \frac{\delta^2 F}{\delta j_\gamma(\mathbf{x})\delta j_\gamma(\mathbf{y})} j_\beta(\mathbf{x}) \right] \\ &= \int d\mathbf{x} \left[G_{\beta\gamma}(\mathbf{x}-\mathbf{y}) j_\gamma(\mathbf{x}) + \phi_\beta(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) - G_{\gamma\gamma}(\mathbf{x}-\mathbf{y}) j_\beta(\mathbf{x}) \right]. \end{aligned}$$

Now let us assume that the initial field was a uniform field in the β-direction, i.e. $j_\alpha(\mathbf{x}) = j\delta_{\alpha,\beta}$, and $\phi_\alpha = \phi\delta_{\alpha,\beta}$. In this case $j_\gamma(\mathbf{x}) = 0$, and we conclude

$$\phi = \int d\mathbf{x} G_{\gamma\gamma}(\mathbf{x} - \mathbf{y})\, j,$$

i.e. in momentum space

$$G_{\gamma\gamma}(\mathbf{q} = 0) = \phi/j.$$

Since we are in the ordered phase, $\phi \longrightarrow const.$ when $j \longrightarrow 0$, and we conclude

$$G_{\gamma\gamma}(\mathbf{q} = 0) = \infty!$$

As $G_{\gamma\gamma}(\mathbf{q} = 0)$ is an even function of $\mathbf{q}$ (by inversion symmetry), the most natural possibility is

$$G_{\gamma\gamma}(\mathbf{q} = 0) \sim 1/q^2.$$

Note that the direction γ can be chosen to be any of the transversal directions to the ordering vector direction β. The corresponding correlation function $G_{\gamma\gamma}(\mathbf{q} = 0) = G_\perp(\mathbf{q} = 0)$ thus described the *transverse correlations*, which we find to be *long-ranged*

$$G_\perp(\mathbf{x}) \sim 1/\left|\mathbf{x}\right|^{d-2}.$$

This argument is completely general. It applies to ANY model with broken continuous symmetry, classical or quantum, and is also valid at any temperature throughout the ordered phase. The excitations associated to these transverse fluctuations are called Goldstone modes, or in quantum systems Goldstone bosons.

Landau theory of the $O(N)$ model

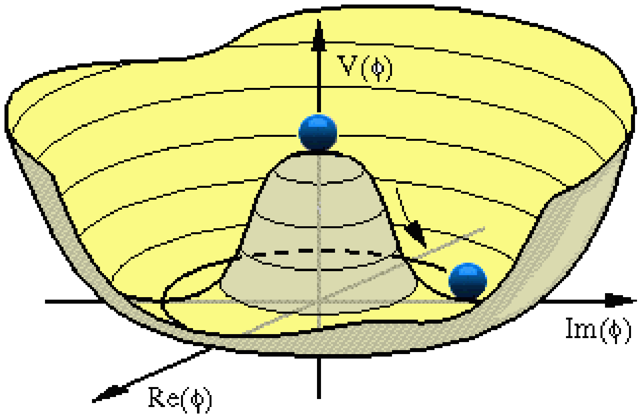
As an example, consider the Landau Action for the $O(N)$ vector field $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$

$$S[\boldsymbol{\phi}] = \frac{1}{2}\int d\mathbf{x}\, \boldsymbol{\phi}(\mathbf{x})\, [r - \nabla^2]\, \boldsymbol{\phi}(\mathbf{x}) + \frac{u}{4}\int d\mathbf{x}\, (\boldsymbol{\phi}(\mathbf{x})\boldsymbol{\phi}(\mathbf{x}))^2 - \int d\mathbf{x}\, \boldsymbol{j}(\mathbf{x})\boldsymbol{\phi}(\mathbf{x})$$
$$= \frac{1}{2}\sum_{\alpha=1}^{N}\int d\mathbf{x}\, \phi_\alpha(\mathbf{x})\, [r - \nabla^2]\, \phi_\alpha(\mathbf{x}) + \frac{u}{4}\sum_{\alpha,\beta=1}^{N}\int d\mathbf{x}\ \phi_\alpha^2(\mathbf{x})\phi_\beta^2(\mathbf{x}) - \sum_{\alpha=1}^{N}\int d\mathbf{x}\ j_a(\mathbf{x})\phi_\alpha(\mathbf{x}).$$

To find the broken-symmetry solution, we minimize this Action, looking for a uniform solution $\boldsymbol{\phi}(\mathbf{x}) = \boldsymbol{\phi}_o = const.$ The corresponding Landau potential

$$V(\boldsymbol{\phi}_o) = \frac{1}{\Omega}S[\boldsymbol{\phi}_o] = \frac{1}{2}r\boldsymbol{\phi}_o^2(\mathbf{x}) + \frac{u}{4}(\boldsymbol{\phi}_o^2)^2.$$

Note that, for $r < 0$ (i.e. $T < T_c$), this potential is spherically symmetric, but it has the form of the "sombrero hat".



The minimum of this potential, instead an isolated point in parameter space, now corresponds to an entire "manifold" of minima, corresponding to the order parameter vector pointing anywhere on the sphere

$$\boldsymbol{\phi}_o = \sqrt{\frac{|r|}{u}}.$$

Physically, this corresponds to the fact that all the spins like to line-up, but in a spherically symmetric model, they are free to choose any direction to do so. Assume that the order parameter takes the form $\boldsymbol{\phi}_o = (\phi_o, 0, \ldots, 0)$, i.e. that it points in direction $\alpha = 1$.

Next, we examine the fluctuations around the symmetry broken state, by expanding the action around this minimum, using the parametrization

$$\boldsymbol{\phi} = (\phi_o + \psi_1, \psi_2, ..., \psi_N).$$

[The deviations ψ_α from the minimum are assumed to be small]. We find

$$\delta S[\boldsymbol{\phi}] = S[\boldsymbol{\phi}] - S[\boldsymbol{\phi}_o] = \int\int d\mathbf{x}\; d\mathbf{y}\;\psi_\alpha(\mathbf{x}) \sum_{\alpha,\beta=1}^{N} \left.\frac{\delta^2 S[\boldsymbol{\phi}]}{\delta\psi_\alpha(\mathbf{x})\delta\psi_\alpha(\mathbf{x})}\right|_{\boldsymbol{\phi}=\boldsymbol{\phi}_o} \psi_\alpha(\mathbf{x}) + O(\psi^4)$$

$$= \frac{1}{2}\int d\mathbf{x}\,\psi_1(\mathbf{x})\,[r{+}3u\phi_o^2 - \nabla^2]\,\psi_1(\mathbf{x}) + \frac{1}{2}\sum_{\alpha=2}^{N}\int d\mathbf{x}\,\psi_\alpha(\mathbf{x})\,[r + u\phi_o^2 - \nabla^2]\,\psi_\alpha(\mathbf{x}) + O(\psi^4).$$

Using the fact that $\phi_o^2 = -r/u$, we conclude that all the "transverse" fluctuations ($\alpha = 2, ...N$) are "massless, i.e.

$$\delta S[\boldsymbol{\phi}] = \frac{1}{2}\int d\mathbf{x}\,\psi_1(\mathbf{x})\,[2u\phi_o^2 - \nabla^2]\,\psi_1(\mathbf{x}) + \frac{1}{2}\sum_{\alpha=2}^{N}\int d\mathbf{x}\,\psi_\alpha(\mathbf{x})\,[-\nabla^2]\,\psi_\alpha(\mathbf{x}) + O(\psi^4).$$

The corresponding correlators take the form

$$G_{11}(\mathbf{q}) = G_{\parallel}(\mathbf{q}) = \frac{1}{2\,|r| + q^2};$$
$$G_{\alpha\alpha}(\mathbf{q}) = G_{\perp}(\mathbf{q}) = \frac{1}{q^2},\ \ \alpha = 2, ...N.$$

Transverse fluctuations - large N approach

The results we have obtained within Landau theory confirms what we expect on general grounds, based on Goldstone's theorem. We found that the transverse fluctuations are "gapless", and give rise to long-range correlations in space. However, this result was obtained on the mean-field level, where the effects of such fluctuations on the physical properties are not addressed. The easiest way to see the effects of the transverse fluctuations is to examine the $O(N)$ theory is the large N limit, where an extended version of mean-field theory becomes exact.

Our starting point is the Landau Action

$$S[\boldsymbol{\phi}] = \frac{1}{2}\int d\mathbf{x}\, \boldsymbol{\phi}(\mathbf{x})\,[r - \nabla^2]\,\boldsymbol{\phi}(\mathbf{x}) + \frac{u}{4N}\int d\mathbf{x}\, \boldsymbol{\phi}^4(\mathbf{x}),$$

where we have rescaled the interaction amplitude $u \longrightarrow u/N$, in order to obtain finite result in the $N \longrightarrow \infty$ limit. We decouple the ϕ^4 term by introducing a collective field σ through a Hubbard-Stratonovich transformation

$$\frac{1}{4N}u\int d\mathbf{x}\, \boldsymbol{\phi}^4(\mathbf{x}) \longrightarrow \frac{1}{2}u\int d\mathbf{x}\, \sigma(\mathbf{x})\boldsymbol{\phi}^2(\mathbf{x}) - \frac{N}{4}u\int d\mathbf{x}\, \sigma^2(\mathbf{x}).$$

The Action now becomes quadratic in the $\boldsymbol{\phi}$-fields

$$S[\boldsymbol{\phi},\sigma] = \frac{1}{2}\sum_{\alpha=1}^{N}\int d\mathbf{x}\, \phi_\alpha(\mathbf{x})\,[r + u\sigma(\mathbf{x}) - \nabla^2]\,\phi_\alpha(\mathbf{x}) - \frac{N}{4}u\int d\mathbf{x}\ \sigma^2(\mathbf{x}).$$

In the large N limit, the partition function can be evaluated by the saddle-point method (as one can see by explicitly integrating out the $\boldsymbol{\phi}$-fields; e.g see Herbut book). The appropriate saddle-point conditions (where $\sigma \longrightarrow \sigma_o$; $\phi_\alpha \longrightarrow \phi_o\delta_{\alpha,1}$)

$$0 = \left.\frac{\delta S[\boldsymbol{\phi},\sigma]}{\delta\phi_\alpha(\mathbf{x})}\right|_{SP};\ \ 0 = \left.\frac{\delta S[\boldsymbol{\phi},\sigma]}{\delta\sigma(\mathbf{x})}\right|_{SP},$$

give

$$0 = \phi_o(r + u\sigma_o);$$
$$\sigma_o = \phi_o^2 + \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{r + u\sigma_o + q^2}.$$

Note that the first condition requires that in the ordered phase ($\phi_o \neq 0$), the "mass" of the transverse propagator $r + u\sigma_o = 0$, i.e. $\sigma_o = |r|/u$. We can immediately eliminate σ_o to calculate the order parameter

$$\phi_o = \left[\frac{|r|}{u} - \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{q^2} \right]^{1/2}.$$

In this expression, the first term is the well-known mean-field expression for the order parameter. The second term comes from the "one-loop" corrections due to the Goldstone modes (transverse fluctuations). Their effect is to reduce the order parameter. Note that we do not find the fluctuation corrections due to the longitudinal mode, since these prove to be of order $1/N$, and thus drop out in the considered $N \longrightarrow \infty$ limit.

We can also calculate the shift of the critical temperature, as follows. At the critical point we can put $\phi_o = 0$, and from $r_c = -u\sigma_o$ we get

$$r_c = -u \int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{q^2}.$$

The corresponding order-parameter exponent $\beta = 1/2$ in all dimensions.

Finally, one can also calculate the susceptibility $\chi = (r + u\sigma_o)^{-1}$ at $T > T_c$, be setting $\phi_o = 0$ in the above equation, and self-consistently calculating σ_o. Note that this equation is identical to the one we have derived earlier, when we considered (self-consistent) perturbation theory in ϕ^4, and can be solved in the same fashion. These integrals, as we have seen, are finite for $d > 2$, but they blow up at $d \leq 2$. We conclude that the singular effect of the Goldstone in $d \leq 2$ is already captured at the level of the self-consistent one-loop (Hartree) approximation, which is exact in the considered large N limit. The resulting large N exponent γ deviates from the Landau prediction $\gamma = 1$, for dimensions $2 < d < 4$.

One more important observation is in order. Even if we consider these one-loop corrections for finite N, where we have to keep the longitudinal mode, we should emphasize that its effects are not very important. This is true since they remain "massive" in the ordered phase, in contrast to the Goldstone (transverse) modes. In particular, the integral

$$\int \frac{d\mathbf{q}}{(2\pi)^d} \frac{1}{2|r| + q^2}$$

is finite in any dimension, provided that $|r| \neq 0$. The longitudinal modes, therfore, do not play an important role in low dimensions.