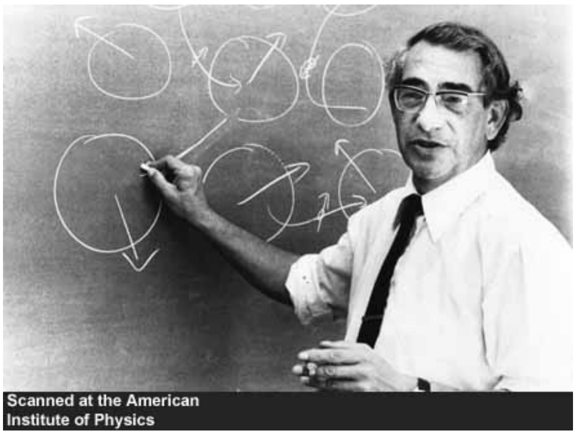
# $\phi^4$ Theory

*The first convincing calculation that demonstrated the validity of the Kadanoff's scaling ideas emerged when Wilson and Fisher developed a renormalization group (RG) approach based on expanding around the upper critical dimension. In this regime, the exact "fixed point " Hamiltonian describing the critical point assumes a particularly simple and tractable form. This allowed to systematically examine the leading fluctuation corrections around Landau theory, and to obtain an exact RG theory in $d = 4 - \varepsilon$ dimensions. A simple formal approach that provides a microscopic basis of this theory is provided by the so-called Hubbard-Stratonovich transformation, which we now describe. It provides a general prescription how to theoretically approach any phase transition involving spontaneous symmetry breaking, and is widely used for classical, and quantum mechanical, clean or disordered systems.*

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### Hubbard-Stratonovich Transformation

We have seen how powerful Landau theory is in providing a general and rather universal description of the critical point. But now we want to beyond it and systematically describe the fluctuation corrections as well. As a simplest example, we again consider the partition function of an Ising model

$$Z = \sum_{\{S_i = \pm 1\}} \exp\left\{ K \sum_{\langle ij \rangle} S_i S_j + \sum_i h_i S_i \right\},$$

the sum $\langle ij\rangle$ is carried over all pairs of nearest-neighbor sites, and we have introduced an arbitrary (not necessarily uniform) external field $h_i$ (as before the temperature $T$ is absorbed in the definitions of the coupling constants $K$ and $h_i$). When we examined the infinite range model we have seen how an elegant "Gaussian" transformation can be used to rewrite the partition function as an integral, which we then solved using a saddle-point method. For the infinite-range model this was particularly simple, since the interaction term in the Hamiltonian was proportional to the square of the magnetization $M = \sum_i S_i$. When the interactions are of short range and we live in finite dimensions the situation is more complicated, but we can still use a generalization of the Gaussian transformation to obtain an integral representation. To do this, we rewrite the interaction term in the following form

$$-\beta H_{int} = \frac{1}{2}\sum_{ij} S_i K_{ij} S_j.$$

Note that the sum now runs over **all sites** $i, j = 1, ..., N$ ($N$ is the number of sites in the lattice) but the matrix elements

$$K_{ij} = \begin{cases} K, & \text{for } i \text{ and } j \text{ nearest neighbors,} \\ 0 & \text{otherwise} \end{cases}.$$

Generalizing the Gaussian trick (**Problem 3.4**), the interaction term can now be **"decoupled"** using **the Hubbard-Stratonovich transformation** to write

$$\exp\left\{\frac{1}{2}\sum_{ij} S_i K_{ij} S_j\right\} = \int \prod_{i=1}^{N} d\phi_i \exp\left\{-\frac{1}{2}\sum_{ij}\phi_i K_{ij}\phi_j + \sum_{ij} S_i K_{ij}\phi_i\right\}.$$

Inserting this expression in the partition function, we get

$$Z = \int \prod_{i=1}^{N} d\phi_i \exp\left\{-\frac{1}{2}\sum_{ij}\phi_i K_{ij}\phi_j\right\} \sum_{\{S_i=\pm 1\}} \exp\left\{\sum_i \left(\sum_j K_{ij}\phi_j + h_i\right) S_i\right\}.$$

Note that the spin-spin interactions are gone, but each spin now experiences a "fluctuating" field $\phi_i + h_i$. The spin sums can now easily be done, and we find

$$Z = \int D\phi \exp\{-S[\phi]\},$$

where we introduced the integration measure $D\phi = \prod_{i=1}^{N} d\phi_i$, and the action (we drop the constant $\ln 2$)

$$S[\phi] = \frac{1}{2}\sum_{ij}\phi_i K_{ij}\phi_j - \sum_i \ln\cosh\left(\sum_j K_{ij}\phi_j + h_i\right).$$

We have succeeded in completely eliminating the spin variables, and rewriting the partition function in the form of a functional integral (in fact nothing but an integral over $N$ integration variables $\phi_i$). So far no approximation has been made!

### Saddle-point solution

Just as in the case of an infinite-range model, we can now attempt a saddle-point solution of this integral, which is the simplest in presence of an uniform field $h_i = h$, when the saddle-point value of the field is also uniform $\phi_i = \phi_o$, and the action takes the form

$$\begin{aligned} S[\phi_o] &= \frac{1}{2}\phi_o^2 \sum_{ij} K_{ij} - \sum_i \ln\cosh\left(\sum_j K_{ij}\phi_o + h\right) \\ &= \frac{N\,K\,z}{2}\phi_o^2 - N\ln\cosh\left(K\,z\,\phi_o + h\right). \end{aligned}$$

The saddle-point conditions $\partial S/\partial\phi_o = 0$ gives

$$\phi_o = \tanh\left(K\,z\,\phi_o + h\right),$$

which is nothing but the familiar Weiss theory solution! Therefore, the saddle-point computation of our functional integral recovers mean-field theory. Of course, for finite range interactions this s only an approximate solution, while in the critical region long-wavelength fluctuations around the saddle point must be included.

### Landau Theory Recovered

Since we are interested in long-wavelength fluctuations, we examine the form of the action assuming that $\phi(\mathbf{x}_i) = \phi_i$ is a smooth function of the lattice coordinate $\mathbf{x}_i$. To obtain a long-wavelength form, we go to momentum space, and we can write

$$\sum_{ij} \phi_i K_{ij} \phi_j = \int \frac{d\mathbf{k}}{(2\pi)^d}\phi(\mathbf{k})K(\mathbf{k})\phi(\mathbf{k}).$$

Here we have used the fact that the field $\phi$ is real, so $\phi(-\mathbf{k}) = \phi(\mathbf{k})$. Furthermore, for short-range interactions on a hypercubic lattice (coordination number $z = 2d$)

$$K(\mathbf{k}) = K\sum_{\alpha=1}^{d}\cos(k_\alpha a)) \approx \frac{1}{2}K\left[z - a^2k^2 + O(k^4)\right].$$

where $a$ is the lattice spacing. Ignoring the momentum dependence coming from the non-linear term lg cosh(...) term, the action takes precisely the Landau-Ginzburg form

$$S[\phi] = \frac{1}{2}\int d\mathbf{x}\, \phi(\mathbf{x})\, [r - \nabla^2]\, \phi(\mathbf{x}) + \frac{u}{4}\int d\mathbf{x}\; \phi^4(\mathbf{x}) - \int d\mathbf{x}\, h(\mathbf{x})\phi(\mathbf{x}).$$

As before, $K_c = z^{-1}$ is the critical coupling at the saddle-point (mean-field) level, we defined $r = (1 - Kz) \approx (K_c - K)/K_c \sim (T - T_c)$; $u = \frac{1}{3}(Kz)^3 \approx \frac{1}{3}(K_c z)^3 = \frac{1}{3}$. We have also chosen to measure the length in units such that $a = z^{-1/2}$, have concentrated on the regime $K \approx K_c$, and have thus replaced $K \longrightarrow K_c$ in the expression for $u$.

We have dropped many terms: all those of order $\phi^6$, $k^2\phi^4$ and many more. We will see shortly that such terms, even if originally kept in the theory, prove to be **irrelevant operators**, i.e. they can be safely ignored in examining the critical behavior. This theory is identical to what the field theorists call the "Euclidean $\phi^4$ field theory", which we shall examine in detail in he following.

The approach we have described is completely general. We can use it whenever the interactions assume a pairwise form, which is almost always the case. Such interaction terms can again be decoupled using the Hubbard-Stratonovich method, and an appropriate Landau action can be derived which, at the saddle-point level, can describe any kind of symmetry breaking. For example, if we have Heisenberg (vector) rather then Ising (up-down, anisotropic) spins, i.e. $\mathbf{S} = (S_1, S_2, S_3)$, the action takes the form

$$S[\phi] = \frac{1}{2}\sum_{\alpha=1}^{3}\int d\mathbf{x}\, \phi_\alpha(\mathbf{x})\, [r - \nabla^2]\, \phi_\alpha(\mathbf{x}) + \frac{u}{4}\int d\mathbf{x}\; \left(\sum_{\alpha=1}^{3}\phi_\alpha^2(\mathbf{x})\right)^2 - \sum_{\alpha=1}^{3}\int d\mathbf{x}\, h_a(\mathbf{x})\phi_\alpha(\mathbf{x}).$$

As we can see in this example, the Landau action has all the terms allowed by symmetry (in this case rotational invariance). In fact, we could have guessed the form of the Landau action based on symmetry arguments, as Landau has originally done.

The application of the Hubbard-Stratonovich method in the quantum case can also be used to describe spontaneous symmetry breaking in itinerant systems, such as the formation of spin and charge density waves. At the saddle point, one recovers the result of the standard Hartree-Fock approximation, but the fluctuations around the saddle point generate corrections. For example, Gaussian fluctuations around the saddle point effectively sum-up all the "ring" diagrams to arrive at the RPA approximation for the electron gas. Another interesting example is found in deriving the BCS mean-field theory for superconductors.

The saddle-point solution of the corresponding action then produces the (original!) Landau-Ginzburg equations describing inhomogeneous superconductors, vortex lattices, etc. Since these applications are the main subject of the "Quantum Many Body Physics - PHY5690" course, we will not elaborate further. Instead, we will concentrate on the physics of the critical region, where the corrections to the saddle point approximation are not small, and the RG methods must be used in a proper theory.