Power Counting, Relevant and Irrelevant Operators

This "power counting" analysis, arising from the renormalization of the Gaussian model, also makes it possible to demonstrate how many possible terms in the effective Hamiltonian ("irrelevant operators") are generically removed by renormalization, explaining the emergence of universality at the critical point. In dimension $d > d_{uc}$ one has to be a bit more careful, due to the emergence of "dangerously irrelevant operators".

obsolete operators

Power counting and stability of the Gaussian model

We continue to examine the theory at the Gaussian level, and examine what happens under rescaling to nonlinear operator

$$S_4 = \frac{u}{4} \int d\mathbf{x}\ \phi^4(\mathbf{x}).$$

Using the same rescaling prescription as before, we conclude

$$u(b) = b^{4-d} u$$

We pause here to fully appreciate the significance of this important result. As we can clearly see from this expression, the "interaction amplitude" u decreases under renormalization for

$$d > d_{uc} = 4.$$

We conclude that the S_4 (quartic) term is irrelevant above the upper critical dimension, and thus the Gaussian theory represents the **exact** fixed point Action in this case. In contrast, the "interaction" S_4 **grows** for $d < d_{uc}$, thus it becomes a **relevant operator**, which cannot be disregarded in the theory.

Irrelevant operators

And what happens to higher order terms, such as those we have dropped in deriving the ϕ^4 theory? For example consider containing the fourth derivative of the form $k^4\phi^2$? After rescaling, such a term is multiplied by b^{-2}. As we can see, such a term can be safely neglected in all dimensions. The same conclusion is true for higher momentum terms of the form $k^{2n}\phi^2$, which scale as $b^{-2(n-1)}$, and are therefore even more irrelevant.

Finally, we examine arbitrary powers of the order parameter, of the form

$$S_m = w_m \int d\mathbf{x}\ \phi^m(\mathbf{x}).$$

Repeating the same analysis, we find

$$w_m(b) = b^{\lambda_w(m)} w,$$

where

$$\lambda_w(m) = m - (m-2)d/2.$$

Such terms are irrelevant above

$$d_{uc}(m) = \frac{2m}{m-2}.$$

As we can see, $d_{uc}(m)$ actually decreases as m grows, and for $m > 4$, $d_{uc}(m) < 4$. We conclude that the terms with $m > 4$ are in fact less relevant that the leading $m = 4$ term, and thus do not modify the upper critical dimension, which remains $d_{uc} = 4$.

Upper critical dimension

The only exception occurs in presence of a ϕ^3(cubic) term ($m = 3$). In this special case we get $d_{uc} = 6$. In magnets such a term is generally absent due to the up-down symmetry of the order parameter. However, in other critical phenomena, such a term may be present, leading to a higher upper critical dimension. Important examples include percolation and spin glass behavior, in both of which $d_{uc} = 6$, precisely for the same reason. Unfortunately, in such cases, the expansion in $\varepsilon = d_{uc} - d$, which is very useful for magnets, proves of little help in physical dimensions $d = 2, 3$, and one is then typically forced to use less controlled real-space RG methods.

But can we get, for example, $d_{uc} = 5$? For thermal critical phenomena we examine here, the answer is NO! This is because, according to the above power-counting argument, this would correspond to $m = 7/2$. This kind of term is generally not allowed for a (bare) Landau functional, since one expect it to be an analytic function of the order parameter. In some cases, such nonanalytic terms can be generated by renormalization, but in the known examples (e.g. Halperin-Lubensky-Ma theory for fluctuation effects in superconductors, gauge-field mediated pairing of composite fermions in quantum Hall systems) this typically leads to a first-order transition, not a critical point with a modified upper critical dimension.

Dangerously irrelevant operators in $d > d_{uc}$

What we said so far may be just a tiny bit misleading. At first glance one may conclude that for $d > d_{uc}$ one may simply drop all non-Gaussian terms, since they scale to zero under renormalization. In specific situations, though, one may need to be a bit more careful, due to the possible presence of so-called dangerously irrelevant operators, which we discuss in the following.

Consider a Gaussian theory at $d > d_{uc}$. We have already shown that in that case the Kadanoff eigenvalues take the values $\lambda_r = 2$, $\lambda_j = 1+d/2$. According to Kadanoff, we should be able to use those and calculate all six critical exponents. For example, $\nu = \lambda_r^{-1} = 1/2$, in agreement with Landau theory. Similarly, $\eta = 0$ within Gaussian theory, so $\gamma = (2-\eta)\nu = 1$, again in accordance with Landau.

The situation is more complicated if we explore the exponents β and δ. Using the exponents relations and Kadanoff theory, in general

$$\beta = \frac{d - \lambda_j}{\lambda_r}; \quad \delta = \frac{2\lambda_j - d}{d - \lambda_j}.$$

Using the values for λ_t and λ_h obtained for the Gaussian model, we find

$$\beta = \frac{d-2}{4}; \quad \delta = \frac{d+2}{d-2}.$$

These values agree with the predictions of Landau theory only for $d = d_{uc} = 4$, but deviate from mean-field results for $d > 4$, where we expect Landau theory to be exact. Something is wrong! But what ?

To understand the source of problems, we go back to the analysis of the scaling formulation, as adapted to the problem at hand. As we have seen, the effective Hamiltonian (i.e. Action) describing our model can be characterized by several parameters: r, j, u, w.... (here w describes some higher order irrelevant operators...). Under rescaling, the Action preserves the same form, but these coupling constants are renormalized. The free energy per unit volume then satisfies a scaling expression of a general form

$$f(r, u, j, w...) = b^{-d} f(b^{\lambda_r} r, b^{\lambda_j} j, b^{\lambda_u} u, b^{\lambda_w} w, ...).$$

We have thus related the free energy of our system at reduced temperature r, field j, and coupling constants u, w..., to the free energy of the same physical system, but at reduced temperature $r(b) = b^{\lambda_r} r$, field $j(b) = b^{\lambda_j} j$, and coupling constants $u(b) = b^{\lambda_u} u$, $w(b) = b^{\lambda_w} w$.... A similar expression is valid for the scaling of the order parameter

$$\phi(r, u, j, w...) = b^{-d+\lambda_j} \phi(b^{\lambda_r} r, b^{\lambda_j} j, b^{\lambda_u} u, b^{\lambda_w} w, ...).$$

For $d > d_{uc}$ we expect the only relevant operators to be r and j (i.e. $\lambda_r > 0$; $\lambda_j > 0$), while all others should be irrelevant, (i.e. $\lambda_u < 0$; $\lambda_w < 0$,...). Therefore, when $b \gg 1$, $r(b)$ and $j(b)$ should grow with b, while $u(b) \longrightarrow 0$, $w(b) \longrightarrow 0$,..., decrease. At sufficiently large b such that $r(b) = b^{\lambda_r} r \sim 1$ (i.e. $b \sim t^{-1/\lambda_r}$) we can write

$$\phi(r, u, j, w...) = r^{(d-\lambda_j)/\lambda_r} f(1, j/r^{\lambda_h \nu}, ur^{-\lambda_u/\lambda_r}, wr^{-\lambda_w/\lambda_r}, ...).$$

For $r \longrightarrow 0$, the parameters $x_u = ur^{-\lambda_u \nu} \longrightarrow 0$; $wr^{-\lambda_w \nu} \longrightarrow 0$, and we are tempted to conclude

$$\phi(r, u, j = 0, w...) \approx r^{(d-\lambda_j)/\lambda_r} \phi(1, 0, 0, 0, ...) \sim r^{(d-\lambda_j)/\lambda_r},$$

with $\beta = (d - \lambda_j)/\lambda_r$, as Kadanoff argued. We have, however, seen that this produces incorrect results for $d > d_{uc}$! Here is what's wrong in applying the naive Kadanoff scaling above the upper critical dimension. After rescaling ($b \gg 1$), the reduced temperature is large, the correlation length is small, and then Landau theory should be perfectly justified to use in any dimension. Thus, we can calculate the zero-field order parameter directly from Landau theory and we find

$$\phi(r(b), j(b), u(b), W(b)...) \approx \sqrt{\frac{|r(b)|}{u(b)}} \sim u(b)^{-1/2}$$

Note that the result indeed does not depend on higher-order coupling constant $w, ...$, but it is a singular (diverging) function of $u(b)$!!! So, instead of assuming that $\phi(1, 0, 0, 0, ...)$ is simply a constant, we need to evaluate it in the limit where $r(b) = b^{\lambda_r} r \sim 1$, and $u(b) \longrightarrow 0$. We get

$$\phi(1, 0, u(b) \longrightarrow 0, 0, ...) \sim u(b)^{-1/2} \sim r^{\lambda_u/2\lambda_r}.$$

We conclude that, because u is a **dangerously irrelevant operator**,

$$\phi(r, u, j = 0, w...) \sim r^{(d-\lambda_j)/\lambda_r + \lambda_u/2\lambda_r},$$

and the exponent

$$\beta = (d - \lambda_j + \lambda_u/2)/\lambda_r.$$

Finally, by using $\lambda_r = 2$, $\lambda_j = 1 + d/2$, and $\lambda_u = 4 - d$, we find

$$\beta = (d - 1 - d/2 + 2 - d/2)/2 = 1/2.$$

As we can see, by properly treating the singularity introducing by the dangerously irrelevant operator, we have recovered the correct Landau result for the order parameter exponent β in the regime above the upper critical dimension. A similar argument (**Problem 4.1**) shows that a proper treatment also recovers the Landau prediction $\delta = 3$, for all $d > d_{uc}$. Of course, for $d < d_{uc}$, the coupling constant u becomes a relevant operator, and grows under rescaling, so the above mechanism does not apply, since it produces a singularity only for $u(b) \longrightarrow 0$. As we will see shortly, for $d < d_{uc}$ the renormalized interaction amplitude $u(b) \longrightarrow u^*$, and the standard Kadanoff scaling is restored. We conclude that the described mechanism applies only for $d > d_{uc}$, where is provides a rigorous justification for the validity of the Landau theory predictions.

Finally, we briefly discuss the hyperscaling relation

$$2 - \alpha = d\nu.$$

According to Landau, $\alpha = 0$, but if we use the above result for $\nu = 1/2$, then the hyperscaling relation would predict

$$\alpha = 2 - d/2.$$

Thus, hyperscaling is violated for $d > d_{uc}$! One may often hear people saying the "hyperscaling is violated because of dangerously irrelevant operators". Such a statement is, in fact, incorrect, and the violation of hyperscaling has a different origin. Note that the above mechanism, related to the dangerously irrelevant operator u is applicable only in a regime where the order parameter is finite, so the ϕ^4 term (proportional to u) in the Landau action must be considered. This is true if we examine the low temperature phase at zero field ($r < 0$, $j = 0$), or the critical region for finite fields ($r = 0$, $j \neq 0$), which is described by the critical exponents β and δ, respectively. In contrast, the specific heat exponent α is well defined even at high temperatures and zero fields ($r > 0$, $j = 0$), where the ϕ^4 term can be disregarded within Landau theory. Thus, the behavior described by $u(b) \longrightarrow 0$ seems of little relevance here. But what's going on?

What happens is that, for $d > d_{uc} = 4$, hyperscaling would suggest $\alpha = 2 - d/2 < 0$. Therefore, what was a singular (diverging) contribution for $d < d_{uc} = 4$, now is not divergent any more, and is a subleading contribution to that given by Landau theory, which predicts a specific heat jump ($\alpha = 0$). Since the proper definition of the exponent α should describe the most singular term, we must conclude that $\alpha = 0$ remains correct in all $d > d_{uc}$, and Landau theory is again proven correct.