Kadanoff Scaling

Even before the formal and mathematically controlled formulation of the renormalization group techniques became available, Kadanoff proposed a physical picture providing a conceptual basis for the scaling behavior. This approach, which later become the standard language of critical phenomena, emphasized that effective coupling constants should be viewed as scale-dependent quantities, giving rise to the ideas of renormalization.

Open questions

We have seen that a remarkable classification of experimental findings around the critical point follows from the Widom's scaling hypothesis, leading to a number of critical exponent relations [the last one follows from the scaling of the correlation function $\chi(R)$ (**Problem 3.1**)]

$$\gamma + 2\beta = 2 - \alpha; \quad \delta = \frac{\gamma}{\beta} + 1; \quad d\nu = 2 - \alpha; \quad \gamma = (2 - \eta)\nu.$$

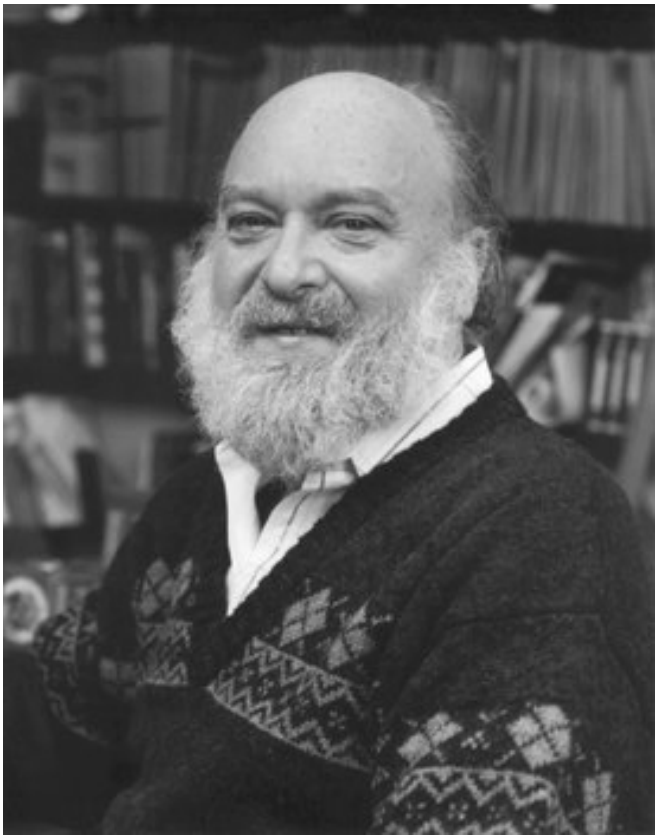
Thus instead of six independent exponents α, β, γ, δ, ν, and η, in fact there are **only two independent parameters**! Even more remarkably, experiments and numerical simulation strongly suggested that their values are **universal**: they only depend on the symmetry of the order parameter and the dimensionality of space, but not on details such as the lattice structure, the range or the precise form of the interactions. But why?

Kadanoff's scaling picture

Early on (1065) Kadanoff made a few reasonable assumptions, which provided a plausible scenario to the mechanism for the critical scaling. Subsequent developments of the renormalization group method (Wilson's work came in 1971) provided rigorous justifications for these assumptions, as we will see shortly. For now, we will take these assumptions as granted, and will explore what the consequences for the critical behavior may be.

We have seen that close to the critical point large fluctuations emerge, with a characteristic spatial scale given by the correlation length $\xi \gg a$ (much longer then the lattice

spacing). If these large distance fluctuations dominate the critical behavior (as suggested by the Ginzburg criterion), then it seems reasonable that the details of the system on short scales (e.g. form of the lattice, or the interaction range) become irrelevant, as the large fluctuations **"wash out"** these details. If this is true, then all systems in the critical region should behave in the qualitatively the same way (i.e. should have same critical exponents), and should be described by the **same effective theory.** Kadanoff early work outlined how such an effective theory should be constructed **in principle**, as follows.



Leo Kadanoff

1. Consider an Ising model on a lattice with spacing a, with Hamiltonian

$$-\beta H = K\sum_{<ij>} S_i S_j + h\sum_i S_i.$$

   We expect a second order phase transition to take place for $K = K_c \sim O(1)$, $h = 0$.

2. Near the critical point, we want to construct an effective theory describing large-scale fluctuations, which we **assume** dominate the critical behavior. To do this, we want

to introduce new variables (new degrees of freedom), describing large "droplets". To do this, we group spins into "blocks" of linear size ba ($b > 1$), thus containing $N_b = b^d$ spins in each block. The block-spin variable

$$\widetilde{S}_k = \zeta \sum_{i_k=1}^{N_b} S_{i_k},$$

where the index i_k runs over all the spins in the k-th block. The constant ζ will remain unspecified; it is typically chosen so that the Hamiltonian retains the same form under rescaling. Note that there are many different configuration of the spins $\{S_{i_k}\}$ that produce the same value of $\widetilde{S}_k$.

3. Now we want to eliminate the short distance fluctuations, by fixing the value of $\widetilde{S}_k$ for each block, but summing in the partition function over all different configurations consistent with this constraint. Formally, we can write

$$Z = \sum_{\{S_i\}} e^{-\beta H[S_i]} = \sum_{\{\widetilde{S}_i\}} e^{-\beta \widetilde{H}[\widetilde{S}_i]},$$

$$e^{-\beta \widetilde{H}[\widetilde{S}_i]} = \sum_{\{S_i\}} \prod_k \delta \left(\widetilde{S}_k - \zeta \sum_{i_k=1}^{N_b} S_{i_k} \right) e^{-\beta H[S_i]}.$$

The procedure we outline here is in principle exact. What is **not clear**, though, is what is the effective Hamiltonian $-\beta \widetilde{H}[\widetilde{S}_i]$ describing the block-spins, after we summed over the internal configurations within each block. In principle, many new terms (e.g. four-spin interactions of the form $K_4 \sum S_{i_1} S_{i_2} S_{i_3} S_{i_4}$) can be generated!

4. We **assume** that we are lucky (or that we started with a correct "bare" Hamiltonian), and that the new Hamiltonian has precisely the same form

$$-\beta \widetilde{H} = K(b) \sum_{<ij>} \widetilde{S}_i \widetilde{S}_j + h(b) \sum_i \widetilde{S}_i,$$

except for new values $K(b)$ and $h(b)$ of the coupling constants. NOTE: if we strictly consider the "bare" theory with Ising spins $S_i = \pm 1$, then the above definition will produce variables $\widetilde{S}_k$ which are not Ising spins any more (they can take many, not two values). However, we temporarily disregard these "details". The essential content of Kandanoff's idea is that if we start with the right variables (not necessarily Ising spins), then the form of the Hamiltonian will remain invariant.

What have we accomplished? Well, the same partition function (i.e. the same free energy) is now calculated from a new lattice model, with lattice spacing $\widetilde{a} = ba$, and new lattice variables $\widetilde{S}_k$. Consider the free energy per unit cell (site) as $f(K,h)$. The **free energy per unit volume** is

$$f_v = a^{-d} f(K,h).$$

At this point note that in general, the **free energy per unit volume** is a physical quantity that can be calculated either from the old or the new lattice model. In contrast, the quantity $f(K,h)$ is a quantity associated with a given lattice model, and thus depends only on the corresponding coupling constants. We get

$$f_v = a^{-d} f(K,h) = (ba)^{-d} f(K_b, h_b),$$

or

$$f(h,K) = b^{-d} f(K_b, h_b).$$

What about the correlation length? Well, the same argument can be used again! If we define the correlation length per lattice spacing as ξ, and the quantity per unit length $\xi_a = a\xi$, then

$$\xi_a = a\xi(K,h) = ba\xi(K_b, h_b),$$

or

$$\xi(K,h) = b\xi(K_b, h_b),$$

Critical point and the β-function

The renormalization group (RG) procedure relates a lattice model with spacing a, and coupling constants K and h, to another lattice model with spacing ba, and coupling constants K_b and h_b. The key assumption of Kadanoff is that both models describe precisely the same physical system. However, each lattice model in question has a critical point at the same $K = K_c$! But the coupling constants K and K_b are not the same! One is closer to K_c then the other. Therefore, and as we can expect, one lattice model is closer to criticality then the other. In particular, the "rescaled" model has a shorter correlation length (since $\xi(K_b, h_b) = \xi(K,h)/b$), and is thus further from criticality.

From the physical point of view this makes perfect sense. Both models describe the same physical system, but since the new model has a larger lattice spacing, then the correlation

length per unit cell is shorter in the new model. What is interesting is that this result is true on both sides of the transition! Since we expect ξ to decrease as we depart from the critical point, we conclude that under rescaling, in RG language we say that under rescaling the coupling constant "flows" away from its critical value.

Next, we examine what happens if K is close to K_c. If K is precisely at K_c, then $K_b = K$; otherwise it moves away. The critical point is identified as a **"fixed point"** of the RG flow. However, if we start close to K_c, then in each iteration we will move by a very small amount. But what is the precise b-dependence? Note that making n iterations is identical as replacing b by b^n. We concentrate on the vicinity of the critical point, where we need many iterations to move away. We can just as well consider a very large b, and we can use the continuum approximation (i.e. think of b as a real number). We want to know by how much does t change as we make one more iteration, i.e. if we change b infinitesimally. We define the "β**-function**"

$$\beta_K(K) = \frac{dK_b}{d\ln b}.$$

The critical point is identified by $\beta(K_c) = 0$, since then K does not change under iteration (i.e. it is a fixed point). Generally, $\beta_K(K)$ is a smooth (analytic) function of K, and we can write

$$\beta_K(K) \approx \lambda_t(K - K_c).$$

To solve the differential equation, we introduce the deviation from the critical point $t = K - K_c$, and we find

$$t(b) = b^{\lambda_t} t.$$

Similarly, in presence of a magnetic field, we get

$$h(b) = b^{\lambda_h} h.$$

Critical exponents

Using these results, we obtain scaling expressions for physical quantities, from which all critical exponents can be obtained. For example, the correlation length satisfies

$$\xi(t, h) = b\xi(b^{\lambda_t} t, b^{\lambda_h} h).$$

The scaling parameter b can be chosen at will, and by specifying $b = t^{-1/\lambda_t}$, $h = 0$, we conclude

$$\xi(t, 0) \sim t^{-1/\lambda_t}.$$

The correlation length exponent is

$$\nu = \frac{1}{\lambda_t}.$$

Similarly, we obtain a scaling expression for the free energy density

$$f(t, h) = b^{-d} f(b^{\lambda_t} t, b^{\lambda_h} h).$$

By choosing again $b = t^{-1/\lambda_t} = t^{-\nu}$, we find

$$f(t, h) = t^{d\nu} f(1, h/t^{\lambda_t \nu}).$$

This is precisely the form identical as the Widom's scaling law, and we read-off the exponents

$$2 - \alpha = d\nu; \quad \Delta = \lambda_h \nu.$$

Note that we have not only provided justification for Widom's free energy scaling hypothesis, but we have also derived the hyperscaling relation, which under phenomenological scaling required an additional assumtion. These result provide our first glimpse to the beauty and power of the RG approach.

What, of course, remains to be done, is to explicitly demonstrate that Kadanoff's assumptions are valid. In the following lecures we examine several specific model calculations where the RG construction can be carried our in technical detail, providing also explicit results for the parameters λ_t and λ_h, thus giving values for all the critical exponents corresponding to a given model.