Critical behavior at the Kosterlitz-Thouless transition

Having derived the RG equations using the mapping to the Sine-Gordon model, we now analyze the RG equations describing the Kosterlitz-Thouless phase transition. We also discuss how essentially the same critical behavior emerges in a number of other physical problems, ranging from the two-dimensional melting, to dissipative two-level systems, and the infamous Kondo problem.

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The RG equations that we have obtained for the Sine-Gordon model can be reinterpreted n terms of the original CG coupling constants $g = 4\pi^2/\widetilde{g}$ and $y_o = a^2\lambda/2$, and the result proves identical to the famous Kosterlitz RG equations

$$\frac{dy_o}{d\ell} = (2 - \frac{\pi}{g})y_o; \qquad \frac{dg}{d\ell} = 4\pi^3 a^4 y_o^2.$$

To simplify the notation, it is convenient to introduce new variables introduce new variables $y = \pi^2 a^2 y_o \sim \exp\{-\beta E_o\}$, $x = g - \pi/2 \sim (T - T_{KT})$, and get

$$\frac{dx}{d\ell} = \frac{4}{\pi}y^2 + O(y^3); \qquad \frac{dy}{d\ell} = \frac{4}{\pi}xy + O(y^2).$$

We immediately notice several important features of these equations, as follows.

1. The vortex fugacity (concentration) y grows under renormalization for $x > 0$, i.e. $T > T_{KT}$. This means vortices unbind

2. The fugacity y decreases under renormalization for $x < 0$, i.e. $T < T_{KT}$. This no free vortices, and the core energy $E_o = -T \ln y_o \longrightarrow +\infty$.

3. The line $y = 0$ on the phase diagram is a fixed line, since there $dx/d\ell = 0$ (i.e. the spin stiffness does not flow).

As we can see the transition is characterized by vanishing (renormalized) fugacity $y^* = 0$. This is why our derivation of the RG equations, which was done perturbatively in y is valid. In fact, and in contrast to the various ε-expansions we have seen before, the presented RG description of the Kosterlitz-Thouless transition is EXACT.

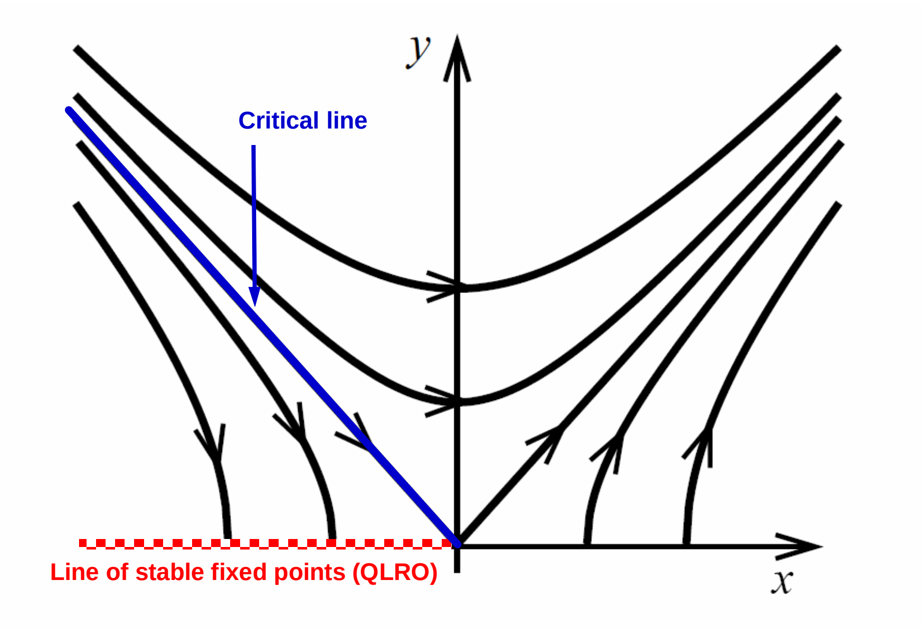


FIG. 1: RG flow diagram for the Kosterlitz-Thouless phase transition, in terms of the reduced temperature $x \sim (T - T_{KT})$, and the fugacity $y \sim \exp\{-\beta E_o\}$. Note the line of stable fixed points, corresponding to the quasi long-range order (QLRO) state (no free vortices). The "separatix" (blue) represents the critical line separating the two phases.

Next, we determine the structure of the flows in more detail. We first observe that

$$\frac{dx^2}{d\ell} = \frac{8}{\pi} x y^2 = \frac{dy^2}{d\ell}.$$

In other words

$$\frac{d}{d\ell}(x^2 - y^2) = 0,$$

i.e.

$$x^2 - y^2 = t,$$

where t is a constant. Thus we have a family of hyperbolae, parametrized by the constant t. For $t = 0$ we get the straight lines

$$y = \pm x.$$

Note that the line $y = -x$, for $x < 0$ is, in fact the separatrix physically corresponding to the critical line separating the two phases. The other branch ($y = x$ for $x > 0$) corresponds to the "relevant direction" describing the high temperature (free vortex) phase. All flows starting "above" the separatrix are "attracted" to this relevant direction. Note that the parameter $t \sim (T - T_{KT})$ thus represent the reduced temperature, since it measures the distance from the critical hypersurface $y = -x$.

Correlation length

Let us calculate the behavior of the correlation length as the transition is approached from the high temperature side. The family of curves describing the free vortex phase correspond to $x^2 - y^2 = t > 0$

$$\frac{dx}{d\ell} = \frac{4}{\pi} y^2 = \frac{4}{\pi} \left(t + x^2 \right).$$

Note that, similarly to other problems at the lower critical dimension, this is a nonlinear differential equation, which generally produces unusual, i.e. non-powerlaw scaling. This differential equation can be easily integrated (table integral) and we find

$$\frac{4}{\pi} \ln b = \frac{1}{\sqrt{t}} \arctan\left(\frac{x}{t}\right) \approx \frac{\pi}{2\sqrt{t}},$$

since $x \longrightarrow \infty$ under scaling. We immediately conclude that the correlation length

$$\xi \sim b \sim \exp\left\{ \frac{\pi^2}{8\sqrt{t}} \right\}.$$

We conclude that, similarly as in the Heisenberg model in $d = 2$, or the Ising model in $d = 1$, the correlation length diverges exponentially as the transition is approached (although we find here $\sqrt{T - T_{KT}}$ in the argument of the exponent).

The free energy per unit volume is generally expected to scale as

$$f \sim \xi^{-d},$$

giving in our case ($d = 2$)

$$f \sim \exp\left\{ -\frac{\pi^2}{4\sqrt{t}} \right\}.$$

In contrast to conventional critical phenomena, this is only a weak "essential" singularity, where all the derivatives vanish. Therefore the specific heat does not diverge at the Kosterlitz-Thouless transition, making it more difficult to observe it experimentally. The specific heat only has a broad "hump" at temperatures somewhat above T_{KT}, where much of the entropy is released as the number of free vortices starts to rapidly drop.

Spin stiffness

A similar argument can be used to determine the critical behavior of the renormalized spin stiffness $K_{eff} = g(b \longrightarrow \infty)^{-1}$ as the KT transition is approached from the low temperature side. Here

$$x^2 - y^2 = |t|\,.$$

Under rescaling $y \longrightarrow 0$, and we find

$$x \approx -\sqrt{|t|}.$$

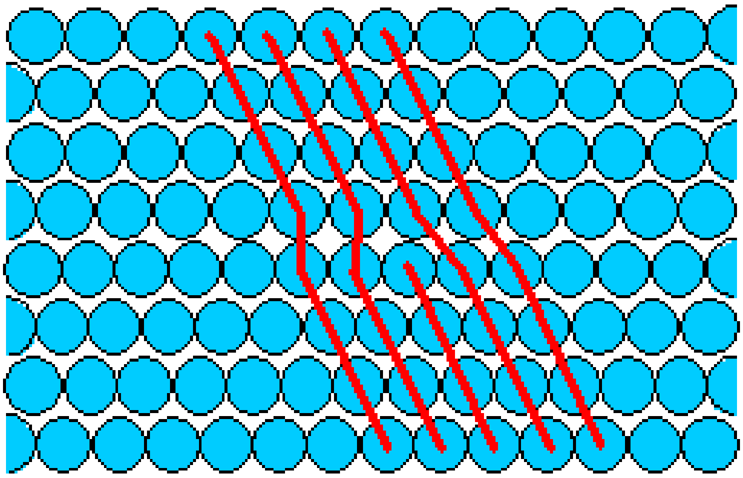
Since

$$x = g - \pi/2 = K^{-1} - \pi/2,$$

we conclude

$$K_{eff} \approx \frac{2}{\pi} + \frac{4}{\pi^2}\sqrt{|t|}.$$

This quantity can be directly measured in superfluids, confirming the prediction of the KT theory. Many other two dimensional systems display phase transitions that belong to the Kosterlitz-Thouless universality class. In all these cases, the phase transition is



Dislocation in a 2D crystal

driven by unbinding of topological point defects. In 2D melting, the topological defects are dislocations. As we can see from the figure, such dislocations can only be formed in pairs, and thus can "annihilate" each other. Similar dislocations are found in smectic *liquid crystals*.

Spin chains with long-range interactions and quantum impurity models

An important class of models that also show phase transitions very similar to the vortex unbinding of Berezinskii, Kosterlitz, and Thouless, include classical spin chains with long range interactions. For example, the Ising model with long range interaction is given by the Hamiltonian

$$H = -J \sum_{ij} \frac{S_i S_j}{R_{ij}^{2-\varepsilon}},$$

R_{ij} is the distance between spins i and j, and $0 < \varepsilon < 1$ is an exponent describing the interaction form. This class of models, as well as the version with $O(N)$ spins can all be treated with RG methods very similar to those used in $d \geq 2$ classical spin models with short range interactions [J. M. Kosterlitz, Phys. Rev. Lett. **37**, 1577 (1976)]. The essential new feature for this class of models (as well as its generalization in $d > 1$) is the fact that the lower critical dimension is modified if the interactions are of long enough range ($\varepsilon \geq 0$). In a sense, the $1/R^2$ spin chains prove to be the equivalent of short range models at their lower critical dimension, and the $O(N)$ models ($N > 1$) have a marginal β-function, and the Ising version shows a KT-like transition. For $\varepsilon > 0$ a regular phase transition is found, where powerlaw behavior near the critical point is found, with critical exponents that depend on ε. These can be calculated using and ε-expansion approach, as presented in the 1976 Kosterlitz paper.

Fro a practical point of view, these long-range spin chains do not have a direct application to actual spin systems, since exchange interactions are generally of short range. However, this class of models proved crucial to understand the behavior of dissipative quantum impurity models, since the dissipative baths generically induce long-range interactions *in time*.

Kondo model

The solution of the Kondo model by Anderson, Yuval, and Hamann (AYH) came historically even before the KT theory, and presented perhaps the first consistent RG calculation as applied to phase transitions. The Kondo model describes a localized magnetic moment such as manganese embedded in a metal such as gold. The Kondo interaction J_K between the Kondo spin and the conduction electrons tends to form a magnetically inert singlet state below the Kondo temperature T_K. This Kondo temperature proves to be an exponential function of microscopic parameters such as the Kondo coupling J_K and the density of states ρ_c for conduction electrons

$$T_K \sim \exp\{-1/\rho_c J_K\}.$$

The theoretical solution of this deceivingly simple problem took a surprisingly long time preceding the seminal AYH work, who succeeded in mapping the behavior of this model onto the solution of an Ising spin chain with $1/R^2$ interactions. In this mapping, the tunneling events where the local moment spin flips due to scattering with conduction electrons, correspond to domain walls of the equivalent Ising chain. In this respect, the problem is similar to ordinary tunneling in of a two-level system. The physically new feature is the fact that conduction electrons represent a dissipative bath, and as a result the tunneling events are not independent. Instead, they feature long-range interactions in time, and the the equivalent Ising chain acquires long-range interactions.

The mapping produced by AYH was achieved by an appropriate discretized path integral representation of the partition function, where the correlations along the spin chains were used to describe the time correlations of the Kondo spin. The appropriate correlation length ξ of the spin chain, thus corresponded to the characteristic correlation time τ_ξ of the Kondo spin, which is inversely proportional to the binding energy of the Kondo singlet

$$\frac{\hbar}{\tau_\xi} \sim T_K.$$

The RG equations describing this model were derived following a procedure similar what Kosterlitz later used for the XY model (indeed following the pioneering AYH work), and take precisely the same form. One interesting difference is the following. Careful calculations show that for the isotropic Kondo problem the bare values of the reduced temperature of the equivalent Ising chain x and the scaled fugacity y prove to lie precisely on the "diagonal" line $x = y$. In addition, the bare value of the reduced temperature x proves proportional to the dimensionless Kondo coupling $g \sim \rho_c J_K$. Thus setting $x = y = g$ in the above RG equations we find

$$\frac{dg}{d\ell} = g^2.$$

Note that this RG equation has precisely the same form as that of the $d = 2$ Heisenberg model (unstable fixed point at $J = 0$), and we find

$$\frac{1}{g(b)} = \frac{1}{g_o} - \ln b,$$

and the correlation length (i.e. the correlation time in the quantum language) is

$$\tau_\xi \sim b \sim \exp\left\{\frac{1}{\rho_c J_K}\right\}.$$

Note that the square root under the exponential is missing, as compared to the standard KT transition result. In the Kondo model this is a result of the special bare values of the coupling constants corresponding to the isotropic Kondo spin. In case of anisotropic coupling of the Kondo spin to the conduction electrons (see the papers by AYH for details) the initial condition relating the bare values of x and y is violated and the standard KT result is obtained.

The phase transition in the Kondo model corresponds to the (bare) Kondo coupling changing sign. For $J_K > 0$ (i.e. antiferromagnetic coupling of the Kondo spin to the conduction electrons), the "fugacity" $y \sim J_K$ is a relevant operator. Physically, this means that the nuber of spin flips proliferates, and the Kondo spin forms a singlet state with the conduction electrons. In the opposite $J_K < 0$ (ferromagnetic Kondo coupling) case , the fugacity (and thus J_K) scales to zero. No spin flips remain, and the spin in "frozen" in time - corresponding to the long range order of the equivalent Ising chain.

In recent years, the mapping of the Kondo-type quantum impurity models to classical spin chains and the associated Coulomb gas representations became a veritable industry, with hundreds of papers published and applied to a variety of models and situations in solid state physics.

Dissipative two-state system

Another interesting application of the long-range spin chain technology as found in examining the dynamics of a tunneling two-level system coupled to a dissipative bosonic bath such as a system of harmonic oscillators. We already discussed the problem of an isolated tunneling center, as described by the Action

$$S[x(t)] = \int_0^{\beta\hbar} dt \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right],$$

with the potential $V(x)$ having two or more minima between tunneling takes place. We have already seen how the $T = 0$ partition function of the problem maps onto an equivalent

$d = 1$ Ising model, and how domain walls of the latter correspond to the tunneling events ("instantons") of the former.

We now couple this particle to a bosonic bath, which is described by a Gaussian action of coupled harmonic oscillators. Since any Gaussian integral can be easily carried out, one is able (as Caldeira and Leggett did) to formally integrate out the harmonic oscillators, resulting to an additional "dissipative" contribution to the Action that takes the form

$$\delta S_\alpha[x(t)] = \alpha \int_0^{\beta\hbar} dt \int_0^{\beta\hbar} dt' x(t) K(t-t') x(t'),$$

where the asymptotic form at long times

$$K(t) \sim \frac{1}{t^{2-\varepsilon}}.$$

A bath with $\varepsilon = 0$ is called an "Ohmic" bath, but one an also consider "sub-Ohmic" baths with $\varepsilon > 0$.

One may study the tunneling processes in presence of such a dissipative bath. As we can see, the problem is mathematically identical to an Ising model with long-range interactions, and one expects a phase transition for sufficiently strong dissipation (corresponding to the interaction J of the Ising chain), where long range order (in time) emerges, corresponding to the *localization* (absence of tunneling) of the quantum particle. From the technical point of view, the problem can be mapped to the Coulomb gas using the instanton approach, or one can use the mapping to the equivalent (dual) Sine-Gordon model. Using the latter method, RG equations can be derived - which again prove to be identical (for $\varepsilon = 0$) to those of Kosterlitz and Thouless, showing how the phase transition belongs to the same universality class. We will not provide further details here. The interested reader is refereed to a nice summary in the lecture notes by Ben Simmons, which also has a list of further applications and references to original works. Interesting recent examples of these dissipative phase transitions is found in the area of "quantum Griffiths phases" emerging as disorder is introduced near quantum critical points. The interplay of quantum criticality, dissipation, and rare even formation, can often lead to puzzling phenomena such as "*Disorder-Induced non-Fermi Liquid Behavior*". For a recent review, see: E. Miranda and V. Dobrosavljevic, *Disorder-Driven Non-Fermi Liquid Behavior of Correlated Electrons,* Reports on Progress in Physics, **68**, 2337–2408 (2005)