Phenomenological Landau Theory

Mean field theory in its original form, or at least if interpreted too literally, ignores all spatial correlations between local degrees of freedoms, as it replaces the environment of a given site by a static external field. However, as first emphasized by Landau, the mean-field approach is easily generalized to spatially non-uniform situations, allowing one to examine the development of spatial correlations as one approaches the critical point.

Inhomogeneous mean-field theory

We have seen that mean-field theory self-consistently calculates the value of the appropriate order parameter across the phase diagram for any given system. its essence is to ignore spatial fluctuations which, as we will see, is often justified if one is not too close to the critical point. A situation where the order parameter is not the same in every point in space, yet mean-field theory should be sufficient is provided in systems where an external field is present, which is a smooth, slowly varying function of the spatial coordinate $\mathbf{x}_i$.

We again concentrate on an Ising ferromagnet on a hypercubic lattice, although the strategy we follow is completely general. In this case, each site in the system still experiences a local field that is proportional to the local magnetization of the neighboring sites, but this local field also slowly varies in space. In the following, we use the notation where instead of the site index i, we use the spatial coordinate $\mathbf{x}_i$ which denotes the position of the i-th site. The local Weiss field acting on a given site $\mathbf{x}$ is

$$h_W(\mathbf{x}) = J\left\langle \sum_{j=1}^{z} S_j \right\rangle = J\sum_{j=1}^{z}\langle S_j\rangle = J\sum_{j=1}^{z} m(\mathbf{x}+\mathbf{a}_i).$$

Here, $\mathbf{a}_i$ represents the lattice vectors connecting the site at $\mathbf{x}$ to each of its z neighbors, of magnitude $|\mathbf{a}_j| = a$, and $m(\mathbf{x}+\mathbf{a}_j)$ is the local magnetization on the neighboring site. The Weiss self-consistency condition now reads

$$m(\mathbf{x}) = \tanh\left[\beta J\sum_{j=1}^{z} m(\mathbf{x}+\mathbf{a}_j) + \beta h(\mathbf{x})\right].$$

We now assume that both the external field and the order parameter vary slowly in space, so we can us the continuum notation and write

$$\sum_{j=1}^{z} m(\mathbf{x}\pm\mathbf{a}_j) \approx \sum_{j=1}^{z}\left[m(\mathbf{x}) \pm \partial_j m(\mathbf{x}) a + \frac{1}{2}\partial_j^2 m(\mathbf{x}) a^2 + O(a^3)\right] = zm(\mathbf{x}) + \frac{1}{2}a^2\nabla^2 m(\mathbf{x}) a^2 + O(a^3).$$

where $\partial_j = \partial/\partial x_j$, and we get

$$m(\mathbf{x}) \approx \tanh\left[\beta J z m(\mathbf{x}) + \frac{1}{2}\beta J a^2 \nabla^2 m(\mathbf{x}) a^2 + \beta h(\mathbf{x})\right].$$

In addition, near the critical point and in presence of a weak external field the order parameter itself is small, and we can expand this expression in powers of $m(\mathbf{x})$

$$-\beta J a^2 \nabla^2 m(\mathbf{x}) + (1-\beta J z)\, m(\mathbf{x}) + \frac{1}{3}(\beta J)^3 m^3(\mathbf{x}) - \beta h(\mathbf{x}) + O(m^5) = 0.$$

This differential equation determines the spatial variations of the order parameter in the critical region, in presence of an arbitrary external field $h(\mathbf{x})$. Such an equation is sometimes called the Landau-Ginzburg equation for order parameter.

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Landau made an interesting observation, which provided a perspective on how general is the validity of such expressions. He noted that such an expression can be obtained from a **variational principle** which determines the equation of state for the order parameter from minimizing a **free energy functional** of the form

$$S[\phi] = \frac{1}{2}\int d\mathbf{x}\, \phi(\mathbf{x})\, [r - \nabla^2]\, \phi(\mathbf{x}) + \frac{u}{4}\int d\mathbf{x}\ \phi^4(\mathbf{x}) - \int d\mathbf{x}\, j(\mathbf{x})\phi(\mathbf{x}).$$

Here, we have used a general notation, where the order parameter $m \longrightarrow \phi$, and we define $j = \beta h$; $r = (1-\beta J z) \approx (T - T_c)/T_c$; $u = \frac{1}{3}(\beta J z)^3 \approx \frac{1}{3}(\beta_c J z)^3 = \frac{1}{3}$. We have also chosen to measure the length in units such that $a = z^{-1/2}$, have concentrated on the regime of temperatures close to $T_c = Jz$, and have thus replaced $T \longrightarrow T_c$ in these expressions. Indeed, setting the variational derivative to zero

$$\frac{\delta S[\phi]}{\delta\phi(\mathbf{x})} = 0$$

gives

$$-\nabla^2\phi(\mathbf{x}) + r\phi(\mathbf{x}) + u\phi^3(\mathbf{x}) - j(\mathbf{x}) = 0.$$

In absence of an external field ($j(\mathbf{x}) = 0$), the free energy is minimized by the uniform solution $\phi(\mathbf{x}) = \phi$, which the minimum of the **potential**

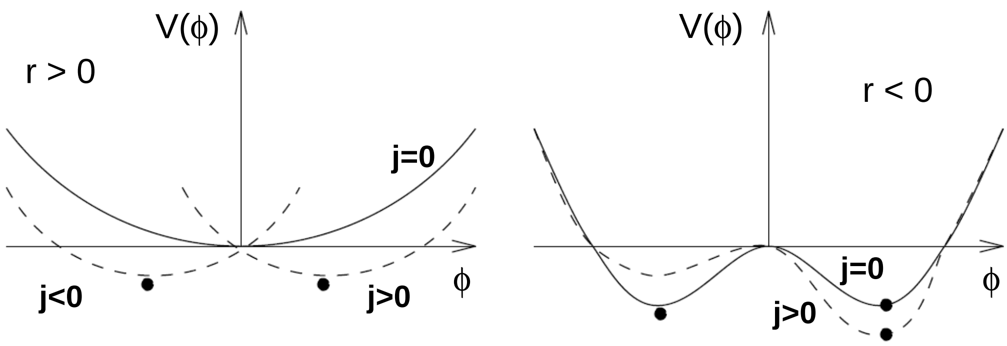
$$V(\phi) = \frac{1}{\Omega} S[\phi(\mathbf{x}) = \phi] = \frac{1}{2} r\phi^2 + \frac{1}{4} u\phi^4 - j\phi,$$

where Ω is the volume of the system.

At $r > 0$ (i.e. $T > T_c$), only a trivial solution $\phi_o = 0$ is found, while for $r < 0$ (i.e. $T < T_c$), additional nontrivial solutions emerge

$$\phi_o = \pm(|r|/u)^{1/2}.$$

But which solution is the correct one at $r < 0$? The answer is provided by the stability

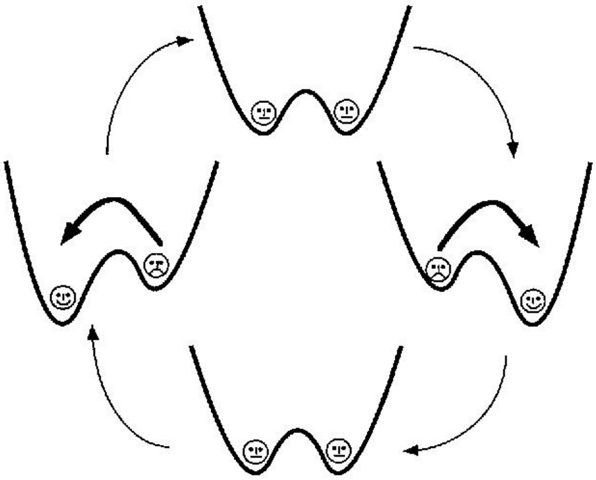


condition requiring the free energy to have a minimum

$$\frac{\delta^2 V(\phi)}{\delta\phi^2} > 0.$$

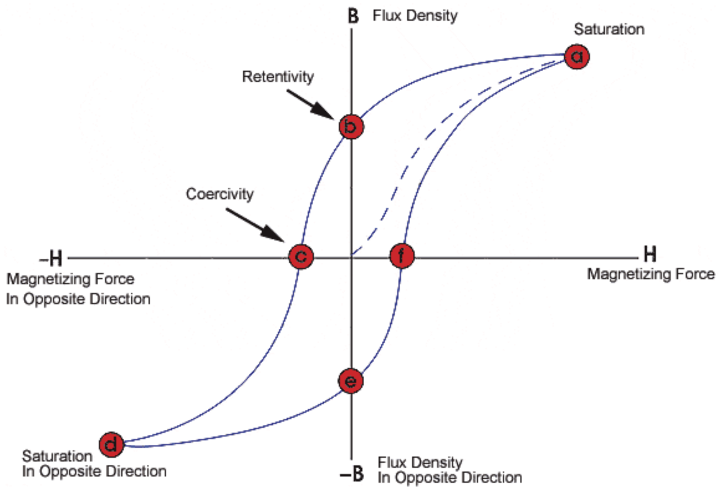
Spontaneous symmetry breaking emerges when the symmetric (trivial) solution $\phi_o = 0$ becomes unstable, which occurs when the "curvature" (coefficient of the quadratic term) of the potential vanishes at $r = 0$. It is easy to check that for $r < 0$, the symmetry broken solutions emerging in the ordered phase are indeed stable, in contrast to the symmetric solution.

For zero external field ($h = 0$), two broken symmetry solutions $\phi_o = \pm(r/u)^{1/2}$ have the same free energy. When a finite field is applied, wither one or the other minimum has lower free energy, and is thus selected as a thermodynamically stable solution. As the



field changes sign from positive to negative, the stable solution "jumps" from a positive to a negative value, corresponding to a first order phase transition. However, if the field is sweeped too fast, then the system can get stuck in a "metastable" state, as in a supercooled liquid below the melting temperature. Eventually, as one goes far enough from the phase transition line, the metastable state becomes unstable. This is called a "spinodal point", where the system "drops out" from the metastable into the thermodynamically stable state. If the process is reversed, then the system can be stuck in the other metastable state, until the other spinodal is reached. Such a process induced by sweeping a magnetic field back and forth leads to the behavior "hysteresis" of ferromegnets. The magnetic field that one needs to apply to reverse the magnetization is called a coercive field, which roughly corresponds to the spinodal point.

In a liquid-gas transition the situation is similar. At $T < T_c$, the free energy has two stable minima with different densities, corresponding to a liquid or a gas. As the first order line is crossed, the two minima exchange stability, and the density "jumps". As the temperature is increased, and we approach T_c, the two minima approach each other and merge at the critical point.



Physical meaning of Landau theory

We have derived the expressions leading to Landau theory using a simple Weiss mean-field theory. However, we note that the resulting free energy functional has a very simple form, essentially includes all terms allowed by symmetry considerations. This is even more apparent if one repeats the construction for models where the order parameter is a vector or a tensor.

In many decades following the early formulations of Weiss and Van der Waals, people have tried to systematically improve the mean-field approximations by adding some effects of fluctuations. This can be done by considering not a single site in an external field but, for example, a cluster consisting of 2,3,4 sites...Theories of this kind have been proposed by Kikuchi, Bethe, Peirls, and many other people. As a result, better estimates for the transition temperature were obtained, in excellent agreement with computer simulations. However, the values of the **critical exponents** resulting from all these formulations were shown to be **identical** as in the simplest Weiss or Landau theory.

To understand this puzzling result, Landau made the following crucial observation. He noted that the expression of the above form can be obtained by simply requiring that the free energy functional is an **analytic function** of the order parameter, consistent with the symmetries of the problem. If this is true, then the free energy can be written as a Taylor expansion of the order parameter ϕ, leading to the given values of the critical exponents, which are **independent** of the numerical values of the coupling constants r, u, etc. These parameters are determined by the specific version of mean-field theory used, but this is

irrelevant as far as the critical exponents are concerned.

Having realized this, Landau was pleased. He was able to explain the apparent **universality** of the critical exponents obtained from many mean-field theories. And for a while people, following Landau, people believed that these results for the critical exponents are indeed exact. Unfortunately, Landau was not quite right. The exponents are **NOT exact**, and to see this we have to go beyond any known mean-field or Landau theory formulations. For Landau's argument to fail, the free energy must be a **nonanalytic** function of the order parameter, which seems very difficult to comprehend. Luckily, the renormalization group theory of Kadanoff and Wilson was able to provide a precise explanation how this phenomenon can take place and why.