Spatial Correlations and the Ginzburg Criterion

Landau theory provides a general mean-field framework which allows the description of spatial correlations in the vicinity of the critical point. In the following we examine its predictions in some detail, which also establishes the limitations the of mean-field theory description.

Spatial correlations

Let us examine the approach to the critical point from high temperatures ($r > 0$), and ask a question how correlated are the spins. In other words, we want to know if the near-by spins are mostly lined up or not, and how this depends on the distance from the critical point. To precisely address this question, we imagine applying an external field only at site $\mathbf{x} = \mathbf{0}$. As a result, a local magnetization $\phi(0) \neq 0$ develops at this site. If the spins are essentially uncorrelated, i.e. do not care about each other, then the magnetization of other spins would vanish, since no external field acts at $\mathbf{x} \neq \mathbf{0}$. On the other hand, if the spins interact strongly, they tend to line up, and so we can expect that $\phi(\mathbf{x}) \neq 0$ in some finite region around $\mathbf{x} = \mathbf{0}$. What actually happens? Well, let us use the formalism of Landau to calculate exactly what goes on in the critical region. According to Landau, the order parameter $\phi(\mathbf{x})$ obeys the following differential Landau-Ginzburg (LG) equation in presence of an arbitrary external field.

$$-\nabla^2\phi(\mathbf{x}) + r\phi(\mathbf{x}) + u\phi^3(\mathbf{x}) - j(\mathbf{x}) = 0.$$

We imagine applying an infinitesimal field at $\mathbf{x} = \mathbf{0}$, and then calculate the magnetization $\phi(\mathbf{x} \neq \mathbf{0})$. In general, for weak fields ("linear response theory") we can write

$$\delta\phi(\mathbf{x},j) = \phi(\mathbf{x},j) - \phi(\mathbf{x},0) = \int d\mathbf{x}'\chi(\mathbf{x}-\mathbf{x}')j(\mathbf{x}') + O(j^2).$$

In the case of interest the field acts only at $\mathbf{x} = \mathbf{0}$, i.e. $j(\mathbf{x}') = j_o\delta(\mathbf{x}')$, and we get

$$\delta\phi(\mathbf{x},j_o) \approx \chi(\mathbf{x})j_o,$$

i.e.

$$\chi(\mathbf{x}-\mathbf{x}') = \left.\frac{\delta\phi(\mathbf{x})}{\delta j(\mathbf{x}')}\right|_{j=0}$$

Thus the nonlocal susceptibility $\chi(\mathbf{x})$ measures the correlations in the system. It is not difficult to show (**Problem 2.3**) that in general

$$\chi(\mathbf{x}) = \langle S(0)S(\mathbf{x})\rangle - \langle S(0)\rangle \langle S(\mathbf{x})\rangle .$$

[The fact that the correlation function $\langle S(0)S(\mathbf{x})\rangle - \langle S(0)\rangle \langle S(\mathbf{x})\rangle$ is equal to the response function $\chi(\mathbf{x})$ is only true in equilibrium. This result is often called the "fluctuation-dissipation theorem").] We now calculate $\chi(\mathbf{x})$ using Landau theory. Starting from the LG equation, we examine the variation introduced by an infinitesimal field applied at $\mathbf{x} = \mathbf{0}$, and keeping only the leading terms in the vicinity of the critical point we find

$$\left(\widetilde{r}-\nabla^2\right) \delta\phi(\mathbf{x}) = j_o\delta(\mathbf{x}'),$$

where

$$\widetilde{r} = r + 3u\phi_o^2; \quad \phi_o = \phi(\mathbf{x}{,}0).$$

Finally

$$\left(\widetilde{r}-\nabla^2\right) \chi(\mathbf{x}) = \delta(\mathbf{x}').$$

We can easily solve this LG equation in momentum space

$$\chi(\mathbf{k}) = \left[\widetilde{r} + k^2\right]^{-1}.$$

The uniform susceptibility is

$$\chi(\mathbf{0}) = \widetilde{r}^{-1} \sim |r|^{\ -1} \sim |T - T_c|^{-1/2},$$

giving the exponent $\gamma = 1$, in agreement with Weiss theory. Here we have used the fact that $\widetilde{r} + 3u\phi_o^2 \sim |r|$, since $\phi_o \approx (|r|/u)^{1/2}$.

The real space expression is obtained by the inverse Fourier transform, and by using the identity

$$\frac{1}{D} = \int_0^\infty dx \exp\{-xD\},$$

and the saddle-point method we find

$$\chi(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{\mathbf{e}^{i\mathbf{k}\mathbf{x}}}{\widetilde{r} + k^2} \sim \exp\{-|\mathbf{x}|/\xi\},$$

were the correlation length

$$\xi \sim |r|^{\ -1/2} \sim |T - T_c|^{-1/2}.$$

Note that $\chi(\mathbf{x})$ is nothing but the Green's function corresponding to the Laplacian, essentially the same problem as that of a free quantum mechanical particle in a potential well with energy $E = -\hbar^2 r/2m$. Because the energy is negative (for $r > 0$), we get what corresponds to a bound state solution, i.e. $\chi(\mathbf{x})$ decays exponentially with distance. The correlation length ξ corresponds to the Bohr radius of the bound state.

We conclude that the spins are correlated (”lined up”) in regions of size ξ, which diverges as the critical point is approached. Therefore, the spin-spin correlations are very large close to the critical point, in contrast to what is assumed by mean-field theory. This result also explains the phenomenon of critical opalescence which results from large density fluctuations near the liquid-gas critical point.

Critical correlations

Exactly at the critical point ($r = 0$), the correlation function takes the form

$$\chi(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{\mathbf{e}^{i\mathbf{k}\mathbf{x}}}{k^2} \sim \frac{1}{|\mathbf{x}|^{d-2}},$$

i.e., long range correlations emerge. Note that the obtained value of the ”anomalous dimension” $\eta = 0$. Away of the critical point, the correlation function assumes the critical (powerlaw) form for $R = |\mathbf{x}| < \xi$, and decays exponentially for $R > \xi$.

We can now calculate the fractal (”Haussdorf”) dimension of the up-spin cluster at criticality. To do this, we assume that an ”up” spin is located at $\mathbf{x} = \mathbf{0}$, and calculate the density (some people call it ”mass”) of up spins in a sphere of radius R around it. This is

$$M \sim \int_o^R dR\, R^{d-1} \chi(R) \sim \int_o^R dR\, R^{d-1} \frac{1}{R^{d-2+\eta}} \sim R^{2-\eta}.$$

We conclude that the fractal (”Haussdorf”) dimension is

$$d_H = 2 - \eta.$$

Since in Landau theory $\eta = 0$ (and is generally very small), we conclude that an almost two-dimensional cluster of up-spins forms in a three dimensional Ising model at criticality. In the ordered phase, of course, bulk magnetization is present, and then $d_H = d$.

Ginzburg criterion

What is the size of the fluctuations as one approaches the critical point? Well, we have seen that for $R < \xi$ the correlation function assumes the critical form

$$\chi(R) = \langle S(0)S(R)\rangle - \langle S(0)\rangle \langle S(R)\rangle \sim R^{-(d-2)}.$$

Given the fact that the crucial fluctuations are uncorrelated at $R \gg \xi$, we can estimate that the *variance* of these fluctuations by evaluating the correlation function at the length scale ξ

$$\chi(R = \xi) \sim \xi^{-(d-2)} \sim (T - T_c)^{d/2-1}.$$

These fluctuations should be compared to the square of the average value

$$\langle S(0)\rangle \langle S(R)\rangle \sim \langle S(0)\rangle^2 \sim (T - T_c).$$

We conclude that the fluctuations dominate if

$$\frac{\langle S(0)S(R)\rangle - \langle S(0)\rangle \langle S(R)\rangle}{\langle S(0)\rangle \langle S(R)\rangle} \sim (T - T_c)^{d/2-2} >> 1.$$

Therefore, the fluctuations will dominate very close to the critical point, provided that

$$d < d_{uc} = 4.$$

We have therefore identified the **upper critical dimensionn** $d_{uc} = 4$ below which **fluctuations dominate** the critical region. In contrast, at $d > d_{uc}$ we conclude that the fluctuations are small, and thus the Landau theory is exact!!!

One has to admit that the above argument is somewhat heuristic. However, we shall examined the effects of fluctuations beyond Landau theory in more rigorous detail shortly, and will be able to estimate very precisely even the size of the critical region where corrections to mean-field theory are important. This is generally a relatively narrow region, explaining why even the simplest Weiss or Van der Waals theories were so successful in explaining many experiments.