Perturbation theory below $d = 4$ and the Ginzburg region

As the problem becomes exactly solvable for $d > 4$, one may guess that just below $d = 4$ the effects of the $u\phi^4$ term may prove to be "small", and one may be able to use a perturbation theory. Let us see what can be learned by straightforward perturbation theory

Perturbation theory in $u\phi^4$

Calculating perturbation theory corrections in $u\phi^4$ is essentially the same problem encountered in quantum field theory for particles interacting through weak two-body forces. Different terms in the perturbation theory can be nicely classified using a graphical representation through "Fenyman diagrams". Since this techniques is already familiar to people familiar with field theory or many-body theory methods we will not elaborate a great deal on technical details, but will instead focus on the physics.

The first calculation that we would like to do in this framework is the evaluation of a leading perturbative correction to the spin susceptibility. We need to calculate

$$\chi = G(k=0) = \int d\mathbf{x}\; e^{i\mathbf{kx}} < \phi(0)\phi(\mathbf{x}) > .$$

At a Gaussian level ($u = 0$), we know that $\chi_o = G_o(\mathbf{k} = 0) = r^{-1}$. We need to "renormalize" the propagator $G(k)$ to leading order in u. In general, one can write

$$G(\mathbf{k}) = [r + k^2 - \Sigma(k)]^{-1},$$

defining the "self-energy" $\Sigma(\mathbf{k})$. We need to calculate the renormalized "mass" $\widetilde{r} = r - \Sigma(0)$. How to do it?

First, calculate the leading correction to the propagator $G(\mathbf{x})$

$$G(\mathbf{x}) =< \phi(0)\phi(\mathbf{x}) >= \frac{1}{Z}\int d\phi\, \phi(0)\phi(\mathbf{x}) \exp\{-S_o - S_{int}\},$$

where (we set $j = 0$)

$$S_o = \frac{1}{2}\int d\mathbf{x}\, \phi(\mathbf{x})\, [r - \nabla^2]\, \phi(\mathbf{x}),$$

and

$$S_{int} = \frac{u}{4}\int d\mathbf{x}\; \phi^4(\mathbf{x}).$$

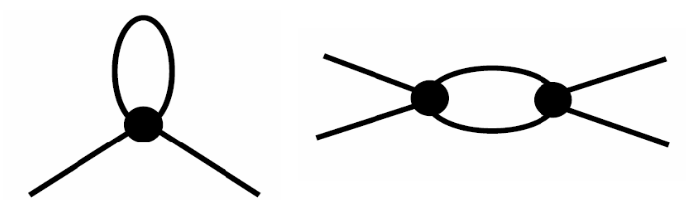

FIG. 1: One loop diagrams that renormalize the "mass" r (left), and the interaction amplitude u (right).

We expand the Boltzmann factor to leading order in u

$$\exp\{-S_{int}\} \approx 1 - \frac{u}{4} \int d\mathbf{x}\ \phi^4(\mathbf{x}),$$

And we get

$$G(\mathbf{x}-\mathbf{y}) = G_o(\mathbf{x}-\mathbf{y}) - \frac{u}{4} \int d\mathbf{x}' \left\langle \phi(\mathbf{x})\phi(\mathbf{y})\phi^4(\mathbf{x}') \right\rangle_o + O(u^2).$$

Here, the expectation value $\langle \cdots \rangle_o$ is taken with respect to the Gaussian action S_o, and the Wick's theorem applies (note the combinatorial factor 12)

$$\left\langle \phi(\mathbf{x})\phi(\mathbf{y})\phi^4(\mathbf{x}') \right\rangle_o = 12 G_o(\mathbf{x}-\mathbf{x}') G_o(0) G_o(\mathbf{x}'-\mathbf{y}).$$

In deriving this result, we have used the "linked cluster theorem" that "disconnected" diagrams cancel out due to the $1/Z$ prefactor. In momentum space we find

$$G(\mathbf{k}) = G_o(\mathbf{k}) + G_o(\mathbf{k})\Sigma(\mathbf{k})G_o(\mathbf{k}) + \cdots,$$

with

$$\Sigma(\mathbf{k}) = -3u \int \frac{d\mathbf{k}'}{(2\pi)^d} G_o(\mathbf{k}') + O(u^2).$$

To compute the shift of the critical temperature, we need to calculate the inverse susceptibility $\chi^{-1} = G^{-1}(k=0)$, and to leading order in u we find (Dyson's equation)

$$G^{-1}(\mathbf{k}) = G_o^{-1}(\mathbf{k}) - \Sigma(\mathbf{k}),$$

giving

$$\chi^{-1} = G^{-1}(k=0) = r + \delta r(r).$$

The critical temperature shift is

$$\delta r(r) = 3u \int \frac{d\mathbf{k}'}{(2\pi)^d} \frac{1}{r + k^2} + O(u^2)$$

It is interesting to look at this result more closely, and examine the behavior of the correction δr as the mean-field critical point is approached. The integrand is spherically symmetric, and we can write

$$\delta r(r) = 3uK_d \int_0^\Lambda dk \frac{k^{d-1}}{r + k^2}$$

where $K_d = S_d/(2\pi)^d$, and S_d is the solid angle in d dimensions (e.g. $S_3 = 4\pi$; $K_3 = (2\pi^2)^{-1} \approx 0.05$). This integral is infrared-convergent at $r = 0$, and thus it simply shifts (down) the critical temperature by a small amount

$$\delta r_o = \frac{3uK_d}{d-2} \Lambda^{d-2}.$$

Assuming that u is small (more precisely to be "self-consistent"), we can replace $r \longrightarrow \widetilde{r}$ under the integral, and we find

$$\begin{aligned} \chi^{-1} = \widetilde{r} &\approx r + 3uK_d \int_0^\Lambda dk \frac{k^{d-1}}{\widetilde{r} + k^2} \\ &\approx r + 3uK_d \int_{\widetilde{r}^{1/2}}^\Lambda dk\, k^{d-3} \\ &= r + \delta r_o - \frac{3uK_d}{d-2} \widetilde{r}^{d/2-1}. \end{aligned}$$

Note that this is a self-consistent equation for $\widetilde{r}(r)$. We can rewrite this as

$$\widetilde{r} + \frac{3uK_d}{d-2} \widetilde{r}^{d/2-1} = r + \delta r_o.$$

Fluctuation correction and the Ginzburg region

We pause to examine in more detail this important result. Note that the fluctuation u-term decreases with $\widetilde{r}$ with a power $d/2 - 1 < 1$ (i.e. it **dominates** for $\widetilde{r}$ small enough), provided that $d < 4$! In this case, the solution takes the form

$$\chi^{-1} = \widetilde{r} \sim \begin{cases} r + \delta r_o, & \widetilde{r} > r^* \\ (r + \delta r_o)^{2/(d-2)} & \widetilde{r} < r^* \end{cases},$$

where the crossover scale (where the two terms are comparable)

$$r^* = \left(\frac{3uK_d}{d-2}\right)^{2/(4-d)}.$$

The susceptibility still diverges at a finite transition temperature given by $\widetilde{r} = 0$, i.e. $r = -\delta r_o$, but the critical behavior is qualitatively modified (exponent $\gamma = 2/(d-2)$). We conclude that the fluctuation corrections cannot be ignored for $d < 4$. The obtained result followed from the leading perturbative correction in the interaction amplitude u. If such a leading term provides a qualitative modifications of the critical behavior, then a naive perturbative treatment is insufficient, and a more systemic theory is required, as provided by the RG approach.

We should note, however, that these corrections are only important close enough to the transition, i.e. within a narrow critical region defined by $\widetilde{r} < r^*$. As an illustration, we can estimate r^* in $d = 3$ using the Weiss theory value $u = 1/3$, and we get $r^* = (K_3)^2 \approx 3\times 10^{-3}$. Farther away from the transition, we see that the fluctuations can be neglected, and the Landau theory remains valid. The same conclusion follows from the RG formulation,as we will discover in the next lecture.

A similar (one loop) calculation provides a correction for the coupling constant $\widetilde{u} = u+\delta u$, with

$$\delta u = 12u^2 K_d \int_0^\Lambda dk \frac{k^{d-1}}{\left(\widetilde{r} + k^2\right)^2} \sim \widetilde{r}^{\,d/2-2}.$$

Again, we conclude that the correction is singular (it grows as $\widetilde{r} \longrightarrow 0$) for $d \leq 4$, and a straightforward perturbative treatment of fluctuation corrections is insufficient. At this stage, it is certainly not clear why we should be allowed to stop at any finite order in perturbation theory. This is why a RG approach must be used, as we discuss in the following.

One final comment is in order. We have noted that the singular corrections arise as infrared divergences, i.e. from long wavelength ($k \longrightarrow 0$) fluctuations. In any approximation that only accounts for short-wavelength fluctuations, a finite infrared cutoff would exist, and all corrections would remain finite. In this case, the qualitative behavior at the critical point would remain unchanged, and all renormalizations can be absorbed in a finite redefinitions of Landau parameters r, u, etc. This argument explains why simple-minded extensions of mean-field theory do not modify the critical behavior, even though their estimates of the critical temperature, for example, can be considerably improved.