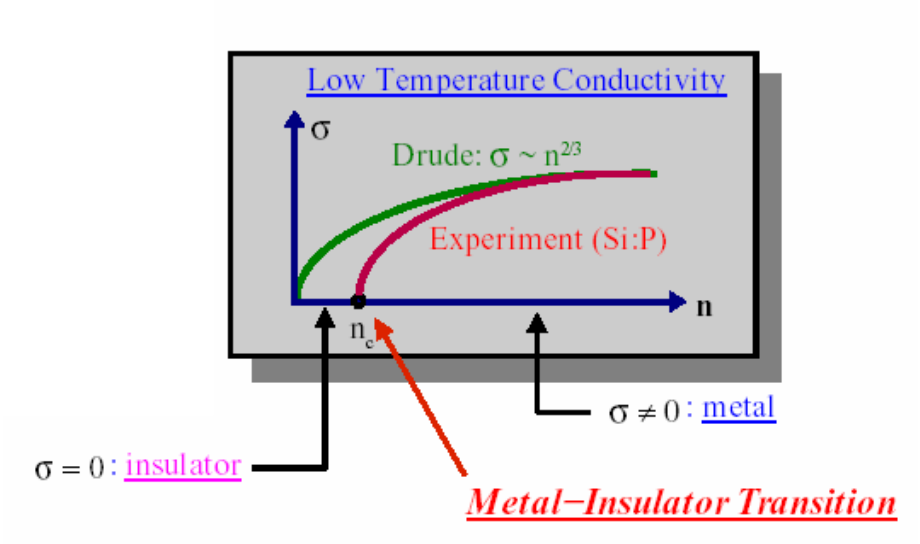
Quantum Critical Scaling

The scaling behavior around critical points is not limited to classical (thermal) phase transitions. Very similar scaling is observed also for quantum critical phenomena occurring at $T = 0$. In many such instances, the microscopic theory is not available. Nevertheless, the scaling hypothesis permits to systematically analyze the experimental data to reveal the details of the critical behavior.



As an example, consider the **metal-insulator transition**. Here, the conductivity plays the role of the order parameter, as it takes a finite value in the metal, but is zero in the insulator. Of course, this is true only at $T = 0$, since activated processes (Mott variable range hopping) produce a small but finite conductivity at finite T even in the insulating phase. Therefore, in the metal

$$\sigma(T) = \sigma_o(n) + mT^{\alpha},$$

where the zero temperature conductivity vanishes at the critical doping level

$$\sigma_o(n) \sim (n_c - n)^{\mu}.$$

In the insulating phase Mott hopping is often observed

$$\sigma(T) \sim \exp\{-(T_o/T)^{1/4}\}.$$

As we can see form these expressions, the distance from the critical doping concentration $\delta n = (n_c - n)/n_c$ plays a role of the Landau parameter. A sharp critical behavior is seen only at $T = 0$, while at any $T \neq 0$, $\sigma(T, n)$ becomes a smooth function of n. Thus, the temperature T plays a role similar to the external field j in the case of a ferromagnet - it "smears" the transition.

In this case the scaling hypothesis postulates that the conductivity can be written as a scaling function of T and δn

$$\sigma(T, n) = T^x f(T/\delta n^{\Delta}).$$

If the exponents x and Δ completely determine the behavior of the conductivity, how do they relate to the conductivity exponent μ? To see this, we note that in order to take the $T \to 0$ limit, and obtain finite conductivity in the metallic phase, the T-dependence from the scaling function has to cancel that from the prefactor. This requires that

$$f(y) \sim y^{-x},$$

and we obtain

$$\sigma(T, n) \sim T^x (T/\delta n^{\Delta})^{-x} \sim \delta n^{-x\Delta}.$$

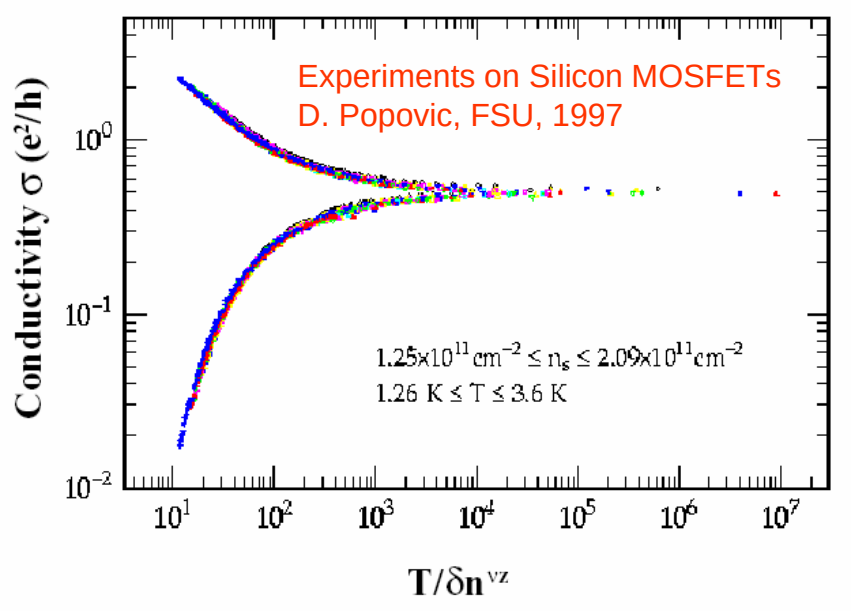
Therefore, scaling requires

$$\mu = x\Delta.$$

How can we extract these critical exponents from the experimental data?

What we have is a series of curves, where the conductivity is measured as a function of temperature for several carrier concentrations (in many cases for each different n one needs to fabricate a new sample). In the following we present a series of steps an experimentalist would take (as does Dragana Popovic in the Magnet Lab) to analyze this critical behavior.

(i) We first have to identify the critical point. We note that only at the critical point the conductivity is a simple power of the temperature (see above). To find the critical concentration, we plot $\sigma(T, n)$ for several n-s as a function of T on a **log-log scale**. The metallic curves will tend to saturate to a finite value at $T = 0$, thus will curve up. The insulating ones have stronger, exponential behavior as a function of T, thus will curve down. Only the critical ($n = n_c$) curve will look like a straight line, and its slope defines the critical



exponent x [For bulk Si:P, experiments find $x \approx 1/2$; for two-dimensional electrons in silicon MOSFETs, $x \approx 0$]. Thus, we have determined n_c, and the critical exponent x.

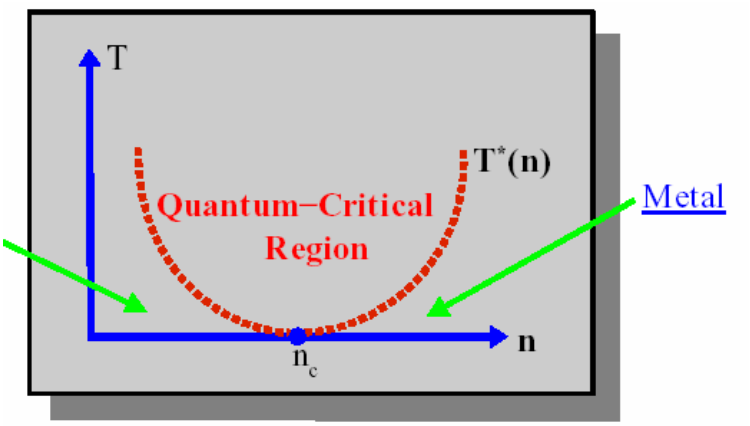
(ii) From the scaling relation, we expect

$$\sigma(T, n)/T^x = f(T/T_o(n)).$$

To extract $T_o(n)$, we plot $\sigma(T, n)/T^x$ on a **log-log** scale as a function of temperature. Now metallic curves all diverge at $T \to 0$, while the insulating ones go to zero. There is only one curve where $\sigma(T, n)/T^x$ is T-independent - the "**separatrix**". Note that if scaling holds, then all these curves must have precisely the same shape and form, except that (on a log-log plot) they will seem "shifted" horizontally. We can pick one concentration on each side of the transition, and then re-plot each curve as a function of T/T_o. Different values of T_o must be tried until the curves perfectly collapse. This will be achieved by choosing a different value $T_o(n)$ for each concentration n. We have thus determined the crossover temperature $T_o(n)$.

Note that all insulating curves collapse on **one branch** of the scaling function, while all metallic ones correspond to the other branch.

(iii) Now we want to extract the exponent Δ. To do this, we plot $T_o(n)$ on a log-log scale as a function of n, and Δ is obtained form fitting the slope. Remarkably, the exponent Δ proves to be exactly the same on both sides of the transition.



Physically, $T_o(n) \sim |\delta n|^{\Delta}$ is the crossover temperature delimiting the **"quantum critical region"**. What is its physical meaning of the exponent Δ? To interpret it, let us think about the dynamics near a critical point. Since large fluctuations (clusters) emerge near a critical point, of characteristic size $\xi \sim \delta n^{-\nu}$, we can expect them to also move slowly, with a characteristic time $\tau \sim \xi^z$. This relation defines the **dynamical critical exponent** z, which relates the scaling of length and time scales. A diverging correlation length thus leads to **"critical slowing down",** a phenomenon generally present in classical and quantum systems. In quantum systems the situation is more complicated. Quantum Mechanics teaches us that due to the Heisenberg Uncertainty Principle, energy and time are related by the expression

$$E \sim \frac{\hbar}{\tau},$$

where τ is a characteristic time. We thus expect that a characteristic temperature (energy) scale emerges near a quantum critical point, given by

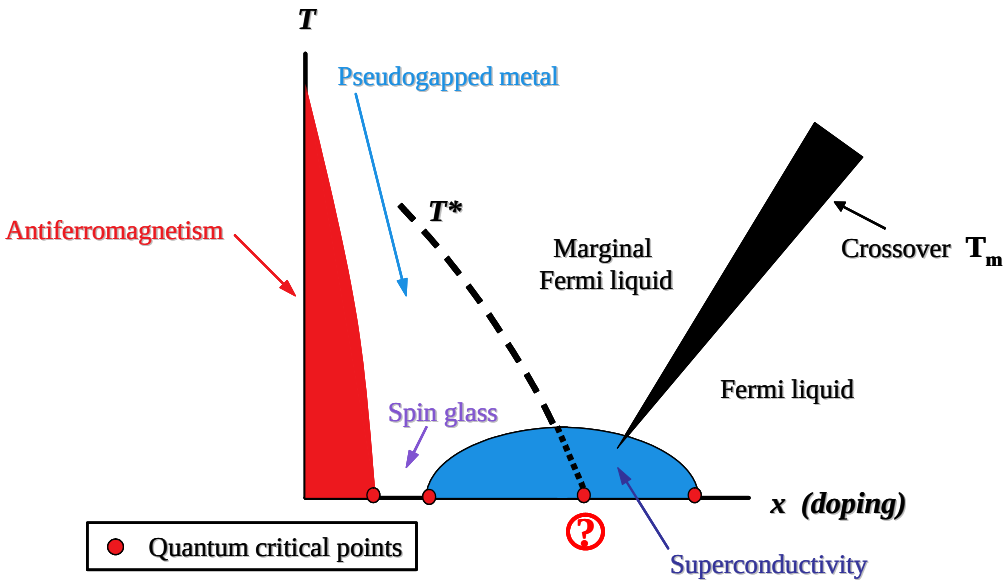
$$T_o \sim \frac{\hbar}{\tau} \sim \xi^{-z} \sim \delta n^{\nu z}.$$

We conclude that

$$\Delta = \nu z.$$

For $T \ll T_o(n)$, and $\delta n > 0$, we expect metallic behavior ($f \to +\infty$). For $T \ll T_o(n)$, but $\delta n > 0$, we expect insulating behavior ($f \to -\infty$). And in the entire region $T \gg T_o(n)$ the behavior is neither that of a metal, not that of an insulator. This is quantum critical behavior, which generally is characterized by powerlaw temperature dependences of all physical quantities. Note that the quantum critical region has "broadens out" as the

temperature is increased. Here the system cannot seem to decide whether to become a metal or an insulator. Only at low temperature the distinction becomes sharp.



Phase diagram of high Tc superconductors

In many systems of current interest, namely the high temperature superconductors (HTS), one is not sure where the relevant quantum critical points (QCPs) may be. One then looks at the characteristic temperature scales emerging from finite temperature data, and tries to identify a point in the phase diagram where such scales seem to vanish in a powerlaw fashion. Such analyses seem to indicate the emergence of a QCP hidden under the superconducting dome. The nature of this QCP is one of the big open questions of modern condensed matter physics, as it may hold the secret of HTS. Recent work discussing this behavior is presented in: "*Self-generated electronic heterogeneity and quantum glassiness in the high temperature superconductors*", by C. Panagopoulos and V. Dobrosavljevic, B **72**, 014536 (2005).