Quasi Long-Range Order in $d = 2$

In models with continuous symmetry, phase (transverse) fluctuations completely eliminate any finite temperature ordering in dimension $d \leq 2$. In $N > 2$ component models, only short range correlations remain at $T > 0$. A more interesting situation is found for the XY ($N = 2$) model, where powerlaw correlations arise in the low temperature region. This exotic state is distroyed by vortices leading to the Kosterlitz-Thouless transition.

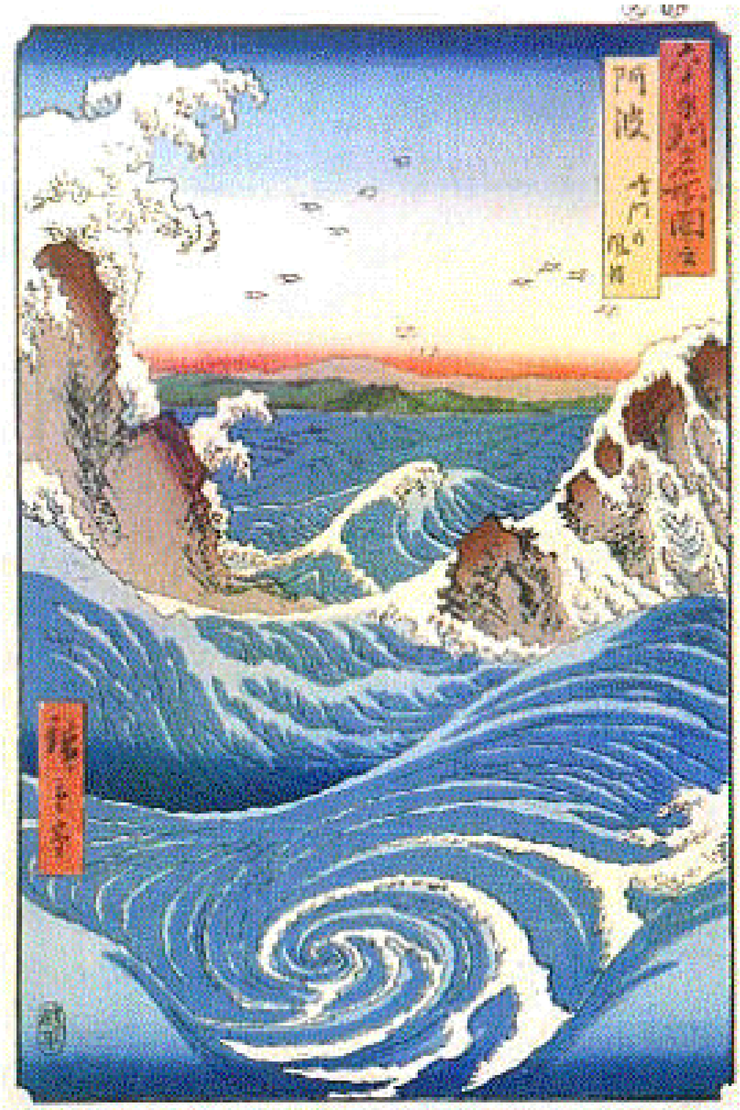

FIG. 1: Whirlpools (vortices) in the woodblock prints by Hiroshige Utagawa

Spatial correlations

All phase transitions we have examined so far corresponded to spontaneous symmetry breaking associated with an appropriate order parameter. Fluctuation effects generally tend to suppressed order, and they prove to be particularly important in low dimensions and models with abundant low energy excitations. For models with continuous symmetry ($N > 2$), we established that no finite temperature ordering exists in dimension $d \leq 2$. However, as we shall see, in a special situation of $N = 2$ (XY) model in $d = 2$, a more intriguing situation is found. Here, an unusual low temperature phase emerges, in which the symmetry is not broken, but correlations assume a long-range form. To characterize this phase, instead of writing an order parameter theory, we turn our attention to the form of the spatial correlation function $G(\mathbf{x})$. In the following we discuss in some detail how it form can be established using standard RG arguments, and discuss how and why an unusual situation emerges for the planar model in $d = 2$.

From general scaling considerations, we expect the spin-spin correlation function to assume the following scaling form

$$G(g, k) = b^{2-\eta} G(g(b), bk),$$

where as before $g = T/\kappa$ is the coupling constant of our $O(N)$ model. If we have a finite temperature transition, then $g^* \neq 0$, and at the critical point $g(b) = g^*$. We can chose $b = k^{-1}$ and we conclude $G_c(k) \sim k^{-2+\eta}$, i.e. $G_c(R) \sim R^{-(d-2+\eta)}$. In the low temperature phase $g(b) \longrightarrow 0$, and we can use spin wave theory to calculate the correlator

$$G(g(b), bk) \approx \frac{g(b)}{(bk)^2},$$

and we conclude $G(k) \sim k^{-2}$, as in Gaussian theory. Physically, at low temperature we have the propagation of essentially free spin waves. Their interactions are *irrelevant operators* in this regime, which is controlled by the $T = 0$ (i.e. $g = 0$) stable fixed point.

In the high temperature phase $g \longrightarrow \infty$, and the RG procedure reduces the computation of $G(k)$ to that of a high temperature model. To be specific, we expect $\delta g(b) \sim b^{1/\nu} \delta g$, and to get out of the critical regime we require $b^{1/\nu} \delta g \sim 1$, i.e. $b \sim \delta g^{-\nu} \sim \xi$. We can thus write

$$G(\delta g, k) \sim \xi^{2-\eta} G_{ht}(\xi k),$$

with $G_{ht}(\xi k)$ being the correlator evaluated at high temperature, viz.

$$G_{ht}(\xi k) \sim \frac{1}{\xi^{-2} + k^2}.$$

We thus expect exponential decay of correlations $G(R) \sim \exp\{-R/\xi\}$, on the scale set by the correlation length.

As we have seen, we generally expect a finite temperature transition for any model in $d > 2$, and the above applies. Indeed, the β function in general takes the form

$$\beta_g = \frac{dg}{d\ell} = -\varepsilon g + \frac{(N-2)}{2\pi} g^2 + O(g^3).$$

Specifically in $d = 2$, the situation is more subtle. Here, the $g = 0$ fixed point becomes unstable for $N > 2$, and the flows turn around and flow to $g \longrightarrow \infty$ for any finite temperature. Using the above argument, we conclude that the correlations become short ranged beyond the scale of the correlation length ξ.

What happens for $N = 2$?! Here, at least to $O(g^2)$, the β-function **vanishes** in $d = 2$! This puzzling result is, in fact more general. As we mentioned before, it reflects the fact that the spin-wave interaction corrections vanish order by order in perturbation theory, and the mechanism that governs the "flow" to $g = \infty$ *does not apply*! Remember, these spin-wave interactions emerge only due to nonlinear processes beyond the Gaussian theory, and which are very different for $N = 2$ and $N > 2$. In contrast, all models looks almost identical at the Gaussian level, which was sufficient to establish the destruction of the ordered phase in $d = 2$.

Because of the "lack of flowing" for the coupling constant g for the $N = 2$ model, we simply use RG to reduce the calculation to that performed at either low or at high temperature. In a very precise sense, this situation is most similar to that at the critical point, where we used the lack of flowing to argue that $G_c(R) \sim R^{-(d-2+\eta)}$. Note, however, that in the the present case we still do not know what is the value of the exponent η. As we shall see, in the $d = 2$ XY model the entire low temperature phase is critical, and the exponent $\eta(T)$ proves to be a nonuniversal (tunable) function of temperature. To calculate it, we need to perform an explicit calculation for the correlator within the Gaussian approximation, as follows.

Gaussian correlator of the $d = 2$ XY model

We concentrate on the XY model, where the spin is a two-component vector $\boldsymbol{\sigma} = (\cos\theta, \sin\theta)$. At low temperatures neighboring spins are almost aligned, and to leading order the Action takes the form

$$S = \frac{1}{2g}\int d^2x\,(\nabla\theta)^2$$

Note that, in contrast to the $N > 2$ models, we are free to extend the integration boundaries from $-\infty$ to ∞, since the Gaussian action does not depend on the absolute value on the angle, only on its gradient. We want to calculate

$$G(\mathbf{x}) = \langle \boldsymbol{\sigma}(0)\boldsymbol{\sigma}(\mathbf{x})\rangle = \langle \mathbf{cos}\,(\theta(\mathbf{x}) - \theta(0))\rangle = \mathrm{Re}\,\langle \exp\{i\theta(\mathbf{x}) - i\theta(\mathbf{x})\}\rangle\,.$$

Let us first try to do the calculation similarly as we did within the ordered phase, assuming that $\theta(\mathbf{x}) - \theta(0)$ is small. If this is true, we can expand

$$\mathrm{Re}\,\langle \exp\{i\theta(\mathbf{x}) - i\theta(0)\}\rangle \approx 1 - \frac{1}{2}\left\langle (\theta(\mathbf{x}) - \theta(0))^2\right\rangle .$$

Since we have a Gaussian Action, this is easy to compute, and we find

$$\left\langle (\theta(\mathbf{x}){-}\theta(0))^2\right\rangle = \frac{g}{(2\pi)^2}\int_o^{2\pi/a}\frac{dk}{k}\int_o^{2\pi} d\phi\,\exp\{ikR\cos\phi\} = \frac{g}{2\pi}\ln(R/C),$$

where C is a constant, and $R = |\mathbf{x}|$. As we can see, the deviation $(\theta(\mathbf{x}){-}\theta(0))^2$ on the average grows with distance! Therefore, although the neighboring spins are almost aligned, the distant spins are not aligned at all, as the spin waves destroy ordering. Therefore, while we are still allowed to use the Gaussian for of the Action, we should not assume that $\theta(\mathbf{x}) - \theta(0)$ is small. In fact, the next term corresponding to $\left\langle (\theta(\mathbf{x}) - \theta(0))^4\right\rangle$ proves to diverge even more strongly, as $\ln^2(R/C)$!

We therefore must go back and evaluate the expression exactly, without expanding the exponent. Luckily, since we have a Gaussian Action this is easy to do just by completing the squares, leading to powerlaw correlations

$$G(R) = \langle \exp\{i\theta(\mathbf{x}) - i\theta(\mathbf{x})\}\rangle = \exp\left\{-\frac{1}{2}\left\langle (\theta(\mathbf{x}){-}\theta(0))^2\right\rangle\right\} \sim R^{-\eta(T)},$$

with the "anomalous dimension"

$$\eta(T) = \frac{g}{2\pi} \sim T.$$

The state of matter we have identified is as close as XY spins can get to order in $d = 2$.