Spontaneous Symmetry Breaking

Second order phase transitions are generally associated with spontaneous symmetry breaking associated with an appropriate order parameter. Identifying symmetry of the order parameter allow one to classify different types of critical phenomena and predict their properties.

Ferromagnet as a paradigm

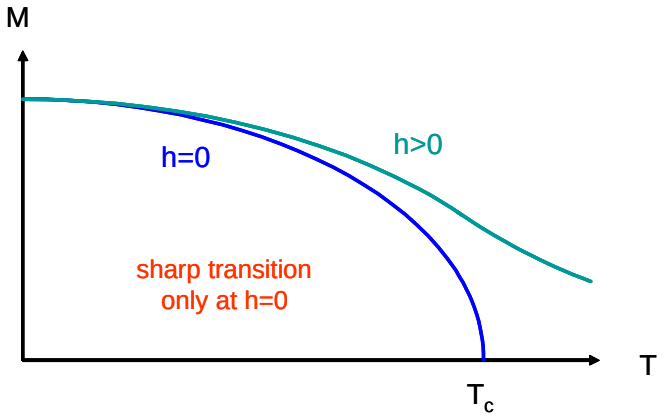
Ferromagnets such as iron or chromium oxide (CrO_2 - studied as we speak in MARTECH for spintronic applications) are also know to everyone as materials from which needles in compasses and even computer hard disks are made of. What is special about a magnet? Its magnetization of course - it makes the magnetic needle turn to the North Pole!

The magnetization M is finite in the low temperature magnetically ordered phase. Simply, in the low temperature ferromagnetic state all the local spins "freeze" in a given direction, which creates a macroscopic magnetic moment. As we heat the system, the thermal fluctuations make the spins "wiggle" around, decreasing the magnetization as the Curie temperature T_c is approached, where it vanishes.

In contrast to the evaporation or melting, the magnetization smoothly (continuously) decreases near the critical point

$$M \sim \pm(T_c - T)^{\beta}.$$

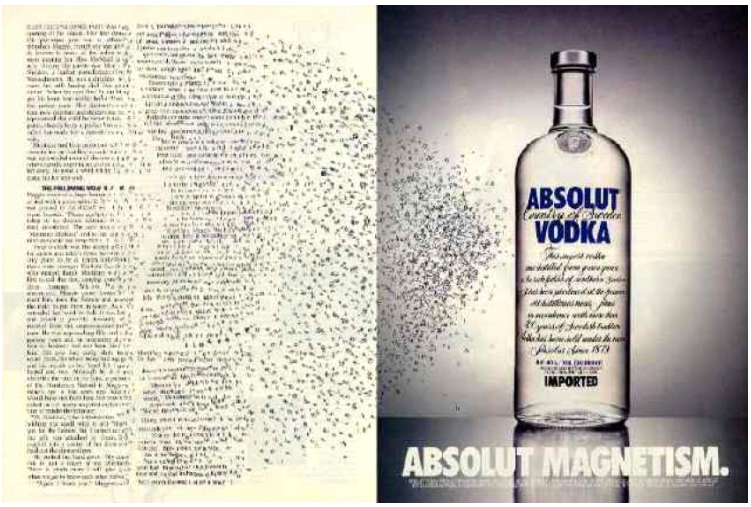
The $\pm$ sign indicates that the direction of the magnetization is not *a priory* known: all the spins want to line up with each other, but in which direction?



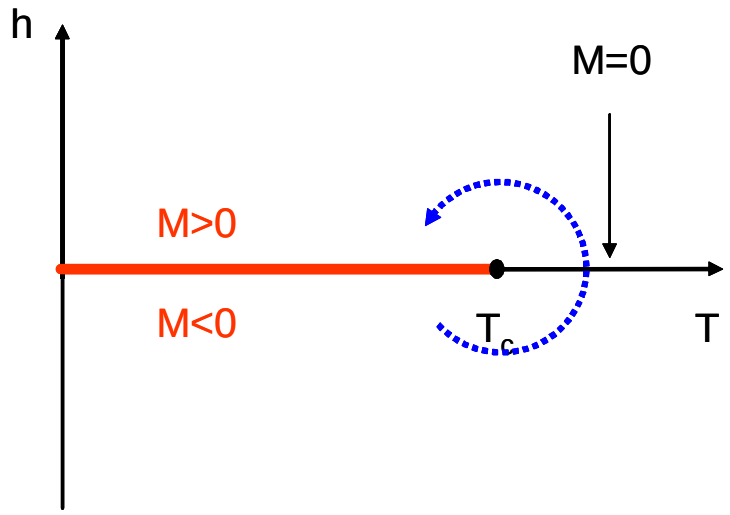
To get ordered, the system has to undergo a spontaneous symmetry breaking: it has to pick a direction. In practice the resulting direction is decided by an infinitesimal external perturbation (magnetic field, or boundary condition). As the system approaches ordering at $h = 0$, and the spins are trying to decide how to order, large fluctuations are found, and the magnetic susceptibility **diverges** as the phase transition is approached from above

$$\chi = \left(\frac{\partial M}{\partial h}\right)_T \sim (T_c - T)^{-\gamma}.$$

A ferromagnet is a prototypical example of a **second order** (continuous) phase transition. The physical processes around second order phase transitions are known as critical phenomena, displaying remarkable **universality**.



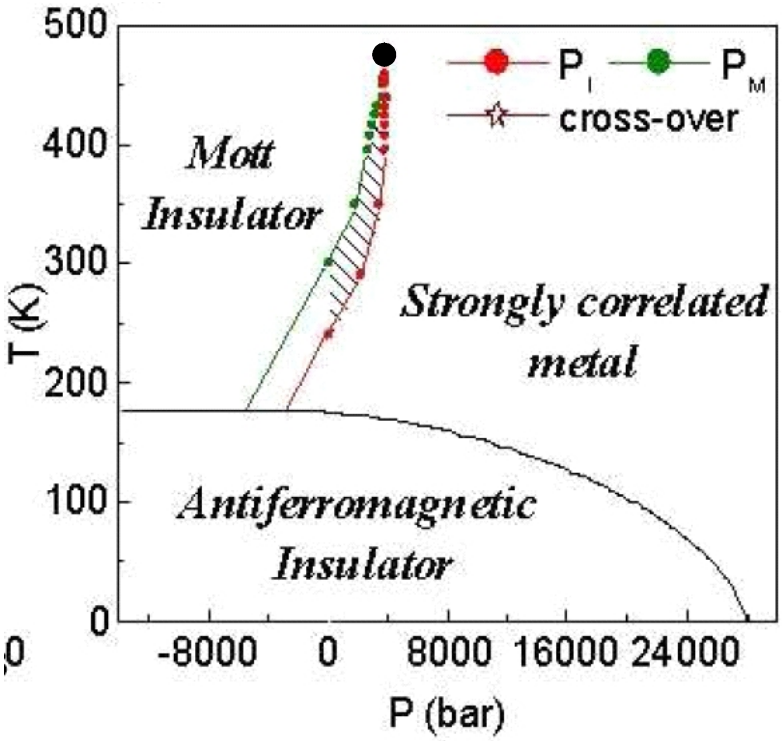
At first glance the situation seems totally different then in the case of the liquid-gas transition. However, the situation is almost the same, and the similarly is observed if one considers the phase diagram of a ferromagnet in the $h-T$ plane (h is the external magnetic field, T the temperature).



The second order phase transition is found only at $h = 0$. In presence of a finite external field, the **symmetry is already broken,** and the magnetization is finite at any temperature. As T is reduced, it gradually grows, and approaches saturation (all spins lined up, no thermal fluctuations) at $T = 0$. Note that, just as around the liquid-gas critical point, we can go "around" the critical point by simultaneously tuning the magnetic field and temperature. But what happens if we consider $T < T_c$, but we sweep the field?

As the field approaches zero from either side, the magnetization jumps! This is a first-order phase transition similar to the evaporation/condensation. The only essential difference is that for ferromagnets the first-order line lies exactly at $h = 0$. But everything else is the same. In fact, for anisotropic (Ising) magnets, the critical exponents are exactly the same as for the liquid-gas critical point. In many ways, the Ising ferromagnet is thus the simplest example of such a critical behavior. Because of this simplicity, we will concentrate much of our attention in this class on this simple model system.

The phenomenon is extremely general. This is because in all these cases the order parameter is a simple **scalar** quantity - it has the same **symmetry**. Other examples are found in very different systems. For example, very recent work on the Cr-doped oxide V_2O_3 has found a similar finite temperature critical point at the end of the first-order line separating



"Universality and Critical Behavior at the Mott transition"
P.Limelette, et al., Science **302**, 89 (2003).

a strongly correlated metal from a Mott insulator. The critical exponents have again found to be identical as for the well-studied liquid gas critical point!

How can a symmetry break spontaneously - ergodicity breaking

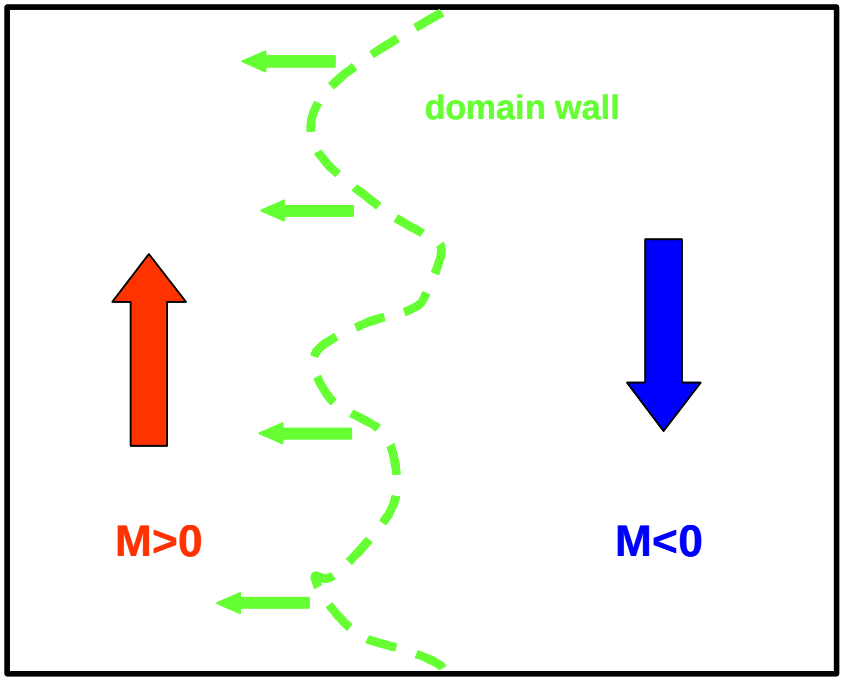
For a ferromagnet in zero field, the up-spin and down-spin states are related by up-down symmetry, and each microscopic configuration has a "partner" of exactly the same energy, where the orientation of all the spins are reversed. Now, according to Boltzmann's ergodic hypothesis, when a system is in equilibrium all the states with the same energy have exactly the same probability, and will therefore be equally populated. For example, if we take two Ising spins $S_i = \pm 1$, $i = 1, 2$ interacting through a ferromagnetic interaction $J > 0$

$$H = -JS_1S_2.$$

Since we have only four possible energy states the partition function can be easily evaluated (**HW # 4**), but we found no phase transition to a ferromagnetic state?! The same argument can be generalized to any finite number of spins - still not transition is found! What is

wrong??!!!

The answer is that our result will be fundamentally different if we have an **infinite** number of spins - if we are in the **thermodynamic limit**. We will not elaborate here the mathematically rigorous basis for this important result. Instead we concentrate on physics. The physical picture behind Boltzmann's ergodic hypothesis is that as time progresses, the system goes from one state to the next, and will eventually visit all possible states. For a system with few degrees of freedom, the transition rate between any two state is appreciable, and thus the system does visit all available states in a short time.



In a large system the situation is very different. Let us assume that we start with a large system where at $t = 0$ all the spins are up. And now we want to estimate the time it takes to reverse all the spins and visit the all-down-spin configuration. The energetically easiest way to achieve this is to "push" a domain wall through a system. the situation is, in this respect, similar as in the problem of nucleation - we have to overcome a free energy barrier! Similarly as in nucleation, the cost of this domain wall is proportional to its surface $\sim L^2$, where L is the **system size**. the big difference is that now L is huge. As we have seen, the probability of reaching this configuration (top of the free energy barrier) is

$$P \sim \exp\{-\sigma L^2/T\}.$$

The time to get there is proportional to this probability and, as we can see, it diverges **exponentially** with the system size!!! Thus as long as L is not truly microscopic this time become astronomically large - essentially infinite (as compare to our short lives, or even the age of the universe).

As a result, the system cannot explore the entire phase space as Boltzmann assumed. It is confined ("stuck") in a subspace of the phase space corresponding to $M > 0$. Therefore, spontaneous symmetry breaking happens **dynamically** - it is a manifestation of **ergodicity breaking**. Does this mean that we cannot use the Boltzmann distribution any more, the ensemble theory, or any of the methods of equilibrium statistical mechanics? Of course not! We simply have to impose a constraint, limiting the statistical sum to the part of phase space the system does explore.

Mean-field theory

We concentrate on a model of an Ising ferromagnet, with the Hamiltonian

$$H = -\frac{J}{2}\sum_{<ij>} S_i S_j - h\sum_i S_i,$$

where $S_i = \pm 1$, and the lattice sum runs over all pairs of nearest neighbor sites on a given lattice, and h is an external magnetic field. The simplest way to obtain the Weiss theory is to replace the fluctuating internal field h_i experienced by a given spin S_i by its thermal average , the "Weiss field"

$$h_i = J\sum_{j=1}^{z} S_j \to h_W = J\left\langle \sum_{j=1}^{z} S_j \right\rangle = Jz\ \langle S_j\rangle = Jz\, m.$$

Here z is the coordination number (number of neighbors to any given site), and m is the magnetization per spin. Now the Hamiltonian reduces to

$$H = (h + Jzm)\sum_i S_i,$$

which corresponds to noninteracting spins in an effective field $h_{eff} = h + Jzm$. The magnetization, therefore, is

$$m = \langle S_i\rangle = \frac{\sum_i S_i \exp\{\beta h_{eff} S_i\}}{\sum_i \exp\{h_{eff} S_i\}} \tanh\{\beta h_{eff}\}.$$

We thus obtain the famous Weiss self-consistency condition

$$m = \tanh\{\beta J z m + \beta h\}.$$

For m small we can expand the right-hand side and obtain

$$m \approx \beta J z m + \frac{1}{3}(\beta J)^3 m^3 - \beta h + \cdots .$$

Itis easy to see that for $h = 0$, this equation has a nontrivial solution only for $T < T_c = Jz$, of the form

$$m \sim (T_c - T)^{1/2}.$$

At $T > T_c$, $m \neq 0$ only for $h \neq 0$, and we get

$$m \approx \frac{h}{T - T_c}.$$

The spin susceptibility

$$\chi = \left.\frac{\partial m}{\partial h}\right|_{h=0} = (T - T_c)^{-1}.$$

We have performed our first calculation of critical exponents, which in this mean-field approximation assume univeral values $\beta = 1/2$; $\gamma = 1$. At this point we do not knwo how justified or robust are these results, or even why they assume such universal values.