Widom Scaling

Remarkable scaling behavior was observed experimentally in the vicinity of critical points much before its theoretical underpinning became available. It allowed to systematically analyze entire families of experimental curves, thus identifying universal features characterizing the critical behavior. A scaling hypothesis was first proposed on phenomenological grounds by Widom, and it imposed significant constraints on the possible forms of the critical behavior. In particular, it established the relations between various critical exponents, thus reducing the number of independent parameters characterizing a critical point.

Mean-field scaling

The simplest form of scaling behavior emerges already within the mean-field description of the critical point, as we now show using very general Landau theory arguments. The key observation of Landau theory is the emergence of the universal critical exponents, which directly follows from the analytic dependence of the free energy on the order parameter. In the following we concentrate on the uniform solution of the LG equations, which take the form

$$r\phi + u\phi^3 = j.$$

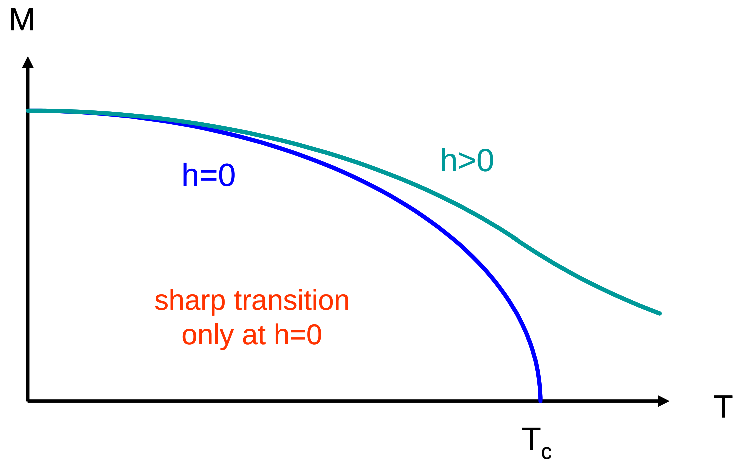
[Remember that the higher powers of ϕ drop out from the calculation near the critical point.] The solution of such an equation describes a family of curves $\phi = \phi(r, j)$. Sharp critical behavior is found only at $j = 0$, where

$$\phi \sim |r|^{1/2}.$$

As r is varied at $j \neq 0$, ϕ is a smooth function of r. Alternatively, one can approach the critical point by tuning the field j at $r = 0$, giving

$$\phi \sim j^{1/3}.$$

Is there a simple formula that captures the entire family of curves? Yes there is, and Landau theory gives us an explicit example, as follows.



Our first observation is that the LG equation has two control parameters (which describe the approach to the critical point at $r = j = 0$). We want to reduce the number of independent parameters, thus we rewrite the LG equations by dividing it with j

$$\frac{r\phi}{j} + \frac{u\phi^3}{j} = 1.$$

We want somehow to eliminate j. We define a new variable $\psi = \phi/j^{1/3}$, or $\phi = j^{1/3}\psi$, and we get

$$\frac{r\psi}{j^{2/3}} + u\psi^3 = 1.$$

Thus, if we also define $x = r/j^{2/3}$, then we can write

$$x\psi + u\psi^3 = 1.$$

This equation has a solution of the form $\psi(x)$, were all the dependence on r and j enters through a single parameter $x = j/r^{3/2}$. Therefore, the general solution of the **LG equation takes the scaling form**

$$\phi(r, j) = j^{1/3}\psi(r/j^{2/3}).$$

What does this means graphically? Well, experimentally we measure ϕ as a function of the reduced temperature r and the reduced magnetic field j. We get a family of curves. Now, if we plot the quantity $\phi/j^{1/3}$ as a function of $r/j^{2/3}$, then **all** curves would **collapse** one on the other! Wow! This means that they all essentially have the same shape, i.e. that all these curves can be described by a single **scaling function** $\psi(x)$.

The only thing missing is the precise form of the function $\psi(x)$. Since the LG equation is a cubic polynomial, this function can actually be calculated in closed form. But this would give a very complicated and not a very illuminating expression. Instead, let us determine the form of this function in three different limits, where its form is very simple.

(a) At $x = 0$, we immediately get

$$\psi(0) = u^{-1/3},$$

and we recover $\phi \sim j^{1/3}$.

(b) At $x \to \infty$, we can drop the $u\psi^3$ term and get

$$\psi(x \to \infty) \approx x^{-1}.$$

Physically, this corresponds to the high temperatures or weak field (more precisely $r \gg j^{2/3}$), where we recover

$$\phi \approx j^{1/3}\left(r/j^{2/3}\right)^{-1} = \frac{j}{r} \sim \frac{h}{T},$$

i.e. Curie's law for free spins.

(c) At $x \to -\infty$, the magnetization is not small, so we can drop $1 \ll x\psi$, $u\psi^3$, and we get

$$\psi \approx \left(|x|/u\right)^{1/2}.$$

Physically, this corresponds to the ordered phase where

$$\phi \approx j^{1/3}\left(|r|/j^{2/3}u\right)^{1/2} \sim |r|^{1/2}.$$

We conclude that our LG theory indeed predicts scaling, but also predicts a specific value of the critical exponents, which may not be correct.

Sometimes it is more convenient to define a variable $y = |x|^{-3/2}$, and define a new scaling function.

$$\varphi(y) = \begin{cases} \psi(y^{2/3}), & x > 0 \\ \psi(-y^{2/3}), & x < 0 \end{cases}.$$

NOTE: Because $-\infty < x < +\infty$, while the new variable $y > 0$, the scaling function becomes multiple-valued, i.e. acquires two branches. We can now obtain an alternative, more elegant scaling form

$$\phi(r, j) = j^{1/3}\varphi(j/r^{3/2}).$$

Alternatively, if we define $\widetilde{\varphi}(y) = y^{-1/3}\varphi(y)$, we can write

$$\phi(r,j) = |r|^{1/2}\,\widetilde{\varphi}(j/r^{3/2}).$$

Widom Scaling

Experiments have been done on many systems, and scaling behavior predicted by Landau theory was found to always hold. On the other hand, the values of the critical exponents proved to be different then mean-field theory predicted. For example, the order parameter still obeys a scaling relation of the form

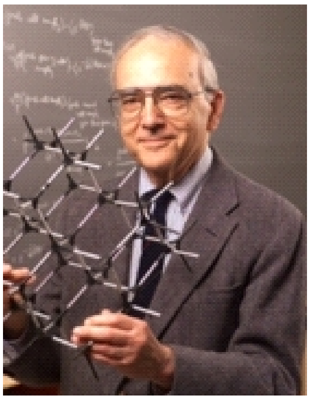
$$\phi(r,j) = j^{1/\delta}\varphi(j/r^{\Delta}),$$

but, for example for the $d = 3$ Ising universality class (liquid-gas critical point), experiments find $\gamma \approx 1.24$, $\beta \approx 0.32$, $\alpha \approx 0.11$, $\delta \approx 4.9$.

Even more mysteriously, experiments seem to suggest that various critical exponents are not really independent, but obey certain constraints. For example, the susceptibility exponent γ, the order-parameter exponent β, and the specific heat exponent α, seem to be related by the following relation (so-called **Rushbrooke's identity**)

$$\gamma + 2\beta = 2 - \alpha.$$

This behavior is consistent with Landau theory, where $\gamma = 1$, $\beta = 1/2$, and $\alpha = 0$, but remarkably, it also holds for the $d = 3$ critical exponents as well!



Ben Widom

A phenomenological scenario that can explain these observations was proposed by Ben Widom in 1965. He proposed that the free energy per unit volume can be written in the scaling form

$$f(r,j) = |r|^{\,2-\alpha} f(j/r^{\Delta}).$$

Various thermodynamic quantities like the magnetization, the susceptibility, and the specific heat, can be obtained from the free energy by taking the appropriate derivatives. For example, since $r \sim (T - T_c)$, the specific heat is

$$C = \frac{\partial}{\partial T^2} f(r,0) = \frac{\partial}{\partial r^2} |r|^{\,2-\alpha} f(0) \sim |r|^{\,-\alpha}.$$

Similarly, the $j = 0$ magnetization

$$\phi(r,j) \sim \frac{\partial f}{\partial j} \sim |r|^{\,2-\alpha-\Delta} f\,'(j/\,|r|^{\Delta}).$$

Therefore, the order parameter exponent

$$\beta = 2 - \alpha - \Delta.$$

Finally, the susceptibility

$$\chi \sim \left.\frac{\partial \phi}{\partial j}\right|_{j=0} \sim |r|^{\,2-\alpha-2\Delta} f\,''(0),$$

and we conclude that the susceptibility exponent

$$\gamma = -2 + \alpha + 2\Delta.$$

Solving for Δ, we obtain the above scaling relation.

We can also compute the exponent δ, as follows. We consider the expression for the magnetization

$$\phi(r,j) \sim |r|^{\,\beta} f\,'(j/\,|r|^{\Delta}),$$

and examine the $r \to 0$ limit. To obtain a finite result, the r dependence from the scaling function has to cancel that of the prefactor. This is only possible if

$$f\,'(y) \sim y^{\beta/\Delta},$$

and we conclude

$$f\,(0,j) \sim j^{1/\delta} \sim j^{\beta/\Delta},$$

or (so called Widom's identity)

$$\delta = \frac{2-\alpha}{\beta} - 1 = \frac{\gamma}{\beta} + 1.$$

Another interesting relation involves the correlation length exponent ν. Near the critical point we expect the correlation length to become large $\xi \sim |r|^{-\nu}$. Within each volume of the order of the correlation length, the spins are essentially aligned, i.e. act like one degree of freedom. **This idea - the notion that the effective number of degrees of freedom varies as the critical point is approached - is the central insight that has motivated the renormalization group approach.**

How many of such regions do we have? Well, if the system size is L, then the entire system has of the order of $(L/\xi)^d$ such regions. The free energy per unit volume may be expected to be proportional to the density of such effective degrees of freedom, thus

$$f \sim \xi^{-d} \sim |r|^{d\nu}.$$

Comparing with Widom's scaling ansatz for the free energy density, we obtain the famous **"hyperscaling" relation**

$$d\nu = 2 - \alpha.$$

Note that, in contrast to the expression relating γ, β, and α, this relation is not satisfied by Landau theory, except in $d = 4$. At first sight this seems strange, since we have argued that Landau theory should produce exact critical exponents in any finite dimension $d > d_{UC}$. The **breakdown of hyperscaling** above the upper critical dimension is a subtle issue, which can be traced down to what is known as **"dangerously irrelevant operators"**, a feature that emerges within the renormalization group description of the problem, to be discussed later.

We conclude that, provided that scaling holds, **only two independent critical exponents exist!**