

**Physics 5645**  
**Quantum Mechanics A**  
**Problem Set X**  
Due: Thursday, Dec 7, 2017

10.1 Commutation relations.

Using the fundamental commutation relations  $[\hat{r}_i, \hat{r}_j] = [\hat{p}_i, \hat{p}_j] = 0$ , and  $[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}$ , show that  $[\vec{L}, \hat{r}^2] = 0$  and  $[\vec{L}, \hat{p}^2] = 0$ , where  $\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$  and  $\hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$ .

10.2 Finite spherical well.

A three-dimensional quantum particle of mass  $m$  is confined by the potential

$$V(r) = \begin{cases} -V_0 & r < a, \\ 0 & r \geq a \end{cases}$$

where  $V_0 > 0$ .

(a) Show that the  $l = 0$  bound states occur when,

$$ka \cot ka = -\rho a,$$

where  $k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$  and  $\rho = \sqrt{\frac{-2mE}{\hbar^2}}$ . (Note the similarity of this problem to Problem 5.3.)

This potential provides a crude approximation to the potential energy of the deuteron (proton-neutron bound state) as a function of proton-neutron separation,  $r$ . In what follows take  $m$  to be the proton-neutron reduced mass ( $m = m_p m_n / (m_p + m_n) \simeq 470 \text{ MeV}/c^2$ ),  $a$  to be the approximate size of the deuteron measured from scattering experiments,  $a = 1.5 \text{ fm}$ , and use the fact that the binding energy of the deuteron, determined from mass measurements, is  $W = 2.23 \text{ MeV}$ .

(b) Determine the value of  $V_0$  in MeV.

(c) Determine whether or not the deuteron has any excited but still bound  $l = 0$  states.

### 10.3 Three-dimensional isotropic harmonic oscillator.

Consider a quantum particle of mass  $m$  moving in the presence of a three-dimensional harmonic potential  $V(r) = \frac{1}{2}m\omega^2 r^2$ . Since the potential is spherically symmetric, we know that energy eigenstates can be taken to be simultaneous eigenstates of  $\vec{L}^2$  and  $L_z$ . The position-space wave functions of these eigenstates will then have the form  $\psi(r, \theta, \phi) = R(r)Y_l^m(\theta, \phi)$ .

- (a) Write down the radial equation for the function  $u(r) = rR(r)$ .
- (b) Introduce the dimensionless radial coordinate  $\rho = \sqrt{\frac{m\omega}{\hbar}}r$  and let  $\epsilon = \frac{2E}{\hbar\omega}$  where  $E$  is the energy of the state and show that the radial equation can be written,

$$\left(-\frac{d^2}{d\rho^2} + \rho^2 + \frac{l(l+1)}{\rho^2}\right)u(\rho) = \epsilon u(\rho). \quad (1)$$

It can be shown that the solutions to (1) have the form  $u(\rho) = e^{-\rho^2/2}v(\rho)$  where

$$v(\rho) = \rho^{l+1}(a_0 + a_2\rho^2 + a_4\rho^4 + \dots) = \rho^{l+1} \sum_{q=0,2,4,\dots}^{\infty} a_q \rho^q.$$

(Feel free to show this yourself, but for purposes of this problem it is OK to just assume it.)

- (c) Show that if  $u(\rho) = e^{-\rho^2/2}v(\rho)$  the radial equation (1) implies that  $v(\rho)$  satisfies the following equation,

$$\frac{d^2v}{d\rho^2} - 2\rho\frac{dv}{d\rho} + (\epsilon - 1)v - \frac{l(l+1)}{\rho^2}v = 0. \quad (2)$$

- (d) Plug in the power series expression for  $v(\rho)$  into (2) and obtain a recursion relation for the coefficients  $a_q$ . Determine the quantized values of  $E$  for which the series truncates.
- (e) Construct the normalized ground state wave function (with  $l = 0$  and  $E = \frac{3}{2}\hbar\omega$ ) and the three first excited state wave functions (with  $l = 1$ ,  $m = -1, 0, 1$  and  $E = \frac{5}{2}\hbar\omega$ ) for this particle.

10.4 Consider a  $\text{He}^+$  ion which consists of a single electron orbiting a nucleus of charge  $+2e$ . If the nucleus of this atom absorbs a positron the nuclear charge will suddenly become  $+3e$  (i.e. the ion will become a  $\text{Li}^{2+}$  ion). Assume the electron in the Helium ion was in its ground state before the positron absorption.

- (a) What is the probability that, immediately after the positron absorption, the electron will be found in the ground state ( $n = 1$ ) of the  $\text{Li}^{2+}$  ion?
- (b) What is the probability that, immediately after the positron absorption, the electron will be found in each of the four degenerate  $n = 2$  excited states of the  $\text{Li}^{2+}$  ion?

You may use the sudden approximation which assumes that the system remains in the ground state of the Hydrogen-like ion immediately after absorbing the positron.

### 10.5 Virial Theorem.

Consider a three-dimensional quantum particle with Hamiltonian,

$$H = \frac{\hat{p}^2}{2m} + V(\hat{r}).$$

- (a) Obtain the Heisenberg equation of motion for the operator  $\Omega = \hat{r} \cdot \hat{p}$ .
- (b) Use your result from Part (a) and the fact that in a stationary state  $\langle \Omega \rangle$  is time-independent to show that if  $|\psi\rangle$  is an eigenstate of  $H$  then

$$\langle \psi | T | \psi \rangle = \frac{1}{2} \langle \psi | \hat{r} \cdot \vec{\nabla} V(\hat{r}) | \psi \rangle,$$

where

$$T = \frac{\hat{p}^2}{2m}$$

is the kinetic energy operator. This is the quantum-mechanical version of the virial theorem.

- (c) Apply your result from Part (b) to the case of a spherically symmetric potential of the form with  $V(r) = kr^\alpha$  and show that in this case the virial theorem states that,

$$\langle T \rangle = \frac{\alpha}{2} \langle V \rangle, \tag{3}$$

in any energy eigenstate.

- (d) Verify that (3) holds for the ground state of the isotropic harmonic oscillator ( $\alpha = 2$ ) and the ground state of the hydrogen atom ( $\alpha = -1$ ).