Physics 5645
Quantum Mechanics A
Problem Set IV

Due: Tuesday, Oct 10, 2017

4.1 Problem 2.2, Sakurai and Napolitano, Pg. 148.

4.2 Useful commutator identities.

(a) Prove the following commutator identity (Eq. 1.6.50e in Sakurai and Napolitano),

$$[A, BC] = [A, B]C + B[A, C].$$

(b) Use this identity, and the fact that $[\hat{x}, \hat{p}_x] = i\hbar$, to prove that

$$[\hat{x}, \hat{p}_x^n] = i\hbar n \hat{p}_x^{n-1}.$$

(c) Using the result of (b), show that

$$[\hat{x}, g(\hat{p}_x)] = i\hbar \frac{dg}{dp_x},$$

for any function g that can be expressed as a power series in its argument. Then show that the three-dimensional generalization of this result is (Eq. 2.2.23a in Sakurai and Napolitano),

$$[\hat{r}_i, G(\hat{\mathbf{p}})] = i\hbar \frac{\partial G}{\partial p_i},$$

where, again, G is a function that can be expressed as a power series in its arguments.

(d) Similarly show that (Eq. 2.2.23b in Sakurai and Napolitano).

$$[\hat{p}_i, F(\hat{\mathbf{r}})] = -i\hbar \frac{\partial F}{\partial r_i},$$

where F is a function that can be expressed as a power series in its arguments.

4.3 Finite translation operator.

(a) Using the results obtained in the previous problem, evaluate the commutator,

$$[\hat{r}_i, T(\mathbf{l})],$$

where

$$T(\mathbf{l}) = e^{-i\hat{\mathbf{p}}\cdot\mathbf{l}/\hbar},$$

is the translation operator for finite spatial displacement.

(b) Using the result of (a), demonstrate how the position expectation value $\langle \hat{\mathbf{r}} \rangle$ changes under translation.

4.4 Consider the precession of the spin of an electron in a uniform magnetic field $\vec{B} = B\hat{k}$ for which the Hamiltonian is

$$H = -\frac{eB}{mc} S_z = \omega S_z,$$

where $\omega = \frac{|e|B}{mc}$.

At time $t = 0$ the electron spin is in the state,

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle,$$

(i.e., an eigenstate of the operator $\hat{\mathbf{n}} \cdot \mathbf{S}$ with eigenvalue $+\hbar/2$ where $\hat{\mathbf{n}}$ is in the xz plane and makes an angle θ with the z axis.)

For parts (a),(b) and (c), work in the Schrödinger picture.

(a) Obtain the time dependent state $|\psi(t)\rangle$.

(b) Find the probability for finding the electron in the $S_x = \hbar/2$ state as a function of time.

(c) Obtain the expectation values of S_x, S_y, and S_z as a function of time. Describe the time evolution of the vector $\langle \mathbf{S} \rangle = \langle \psi(t)|\mathbf{S}|\psi(t)\rangle$.

For parts (d) and (e), work in the Heisenberg picture.

(d) Write down the Heisenberg equations of motion for the time-dependent operators $S_x(t)$, $S_y(t)$, and $S_z(t)$. Solve these equations to obtain expressions for $S_x(t)$, $S_y(t)$, and $S_z(t)$. [Your expressions should be in terms of t and $S_x(0)$, $S_y(0)$, and $S_z(0)$.]

(e) Using the results of Part (d), again obtain the expectation values of S_x, S_y, and S_z as a function of time. Describe the time evolution of the vector $\langle \mathbf{S} \rangle = \langle \psi(0)|\mathbf{S}(t)|\psi(0)\rangle$ and show that it is the same as that obtained in Part (c) using the Schrödinger picture.

4.5 Time Evolution of free particle wave packet (Schrödinger picture).

Consider the motion of a free particle in one-dimension with Hamiltonian

$$H = \frac{\hat{p}^2}{2m}.$$

At time $t = 0$ the particle is in a state $|\psi(0)\rangle$ with position-space wave function

$$\langle x|\psi(0)\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp^{-x^2/(2d^2)} e^{ikx}.$$

(a) Working in the Schrödinger picture, find the position space wave function at time t.

Answer:

$$\langle x|\psi(t)\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \frac{\exp^{-(x-v_g t)^2/\left[2d^2\left(1+i\frac{\hbar}{md^2}t\right)\right]}}{\sqrt{1+i\frac{\hbar}{md^2}t}} e^{ik(x-v_p t)}, \tag{1}$$

where $v_g = \hbar k/m$ is the group velocity, and $v_p = \hbar k/(2m)$ is the phase velocity.

(b) Specializing to the case $k = 0$, compute the expectation values $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{p}^2\rangle$. Determine $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ as a function of time and verify that the uncertainty principle always holds.