Physics 5645
Quantum Mechanics A
Problem Set V

Due: Tuesday, Oct 24, 2017

5.1 Time evolution of free particle wave packet (Heisenberg picture).

Consider once more our good friend the Gaussian wave packet with position-space wave function,

$$\langle x|\psi\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} e^{-x^2/(2d^2)}. \tag{1}$$

(a) Prove that for this state $\hat{x}|\psi\rangle = i(\text{real quantity})\hat{p}|\psi\rangle$.

Now, working in the Heisenberg picture, consider the time evolution of this state for the free-particle Hamiltonian,

$$H = \frac{\hat{p}^2}{2m}.$$

Assume that at time $t = 0$ (used to define the Heisenberg picture) the particle is in the state $|\psi\rangle$ given in (1).

(b) Write down the Heisenberg equations of motion for $\hat{x}(t)$ and $\hat{p}(t)$. Solve them to obtain expressions for these operators in terms of $\hat{x}(0)$, $\hat{p}(0)$, and t.

(c) Obtain $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ as a function of time for this particle and verify that the uncertainty principle is always satisfied.

In doing Part (c) you may use the following expectation values,

$$\begin{aligned} \langle\psi(0)|\hat{x}(0)|\psi(0)\rangle &= 0, \qquad \langle\psi(0)|\hat{p}(0)|\psi(0)\rangle = 0, \\ \langle\psi(0)|\hat{x}^2(0)|\psi(0)\rangle &= d^2/2, \qquad \langle\psi(0)|\hat{p}^2(0)|\psi(0)\rangle = \hbar^2/(2d^2), \end{aligned}$$

since you've calculated them before. If done properly the calculation for Part (c) should not be lengthy. You may find the result of Part (a) useful.

5.2 Obtain the energy levels and normalized position-space wave functions for the energy eigenstates of an infinite square well of width L centered at $x = L/2$. If the particle is in the state $\psi(x) = Ax(x - L)$ at time $t = 0$, obtain an expression for the probability to find the system in any of the energy eigenstates.

5.3 Consider a particle in the finite square well potential

$$V(x) = \begin{cases} 0, & -a \le x \le a \\ V_0, & \text{otherwise} \end{cases}$$

where $V_0 > 0$.

Because $V(x) = V(-x)$ we know that the bound state solutions will have either even or odd parity and so must have the forms,

	Even Parity	Odd Parity
$x \le -a$	$\psi(x) = Ae^{\rho x}$	$\psi(x) = Ae^{\rho x}$
$-a < x < a$	$\psi(x) = B\cos(kx)$	$\psi(x) = B\sin(kx)$
$x \ge a$	$\psi(x) = Ae^{-\rho x}$	$\psi(x) = -Ae^{-\rho x}$

where $k = \sqrt{2mE/\hbar^2}$ and $\rho = \sqrt{2m(V_0 - E)/\hbar^2}$.

In class we showed that even parity bound states can be found by solving the equation

$$\tan z = +\sqrt{\left(\frac{z_0}{z}\right)^2 - 1},$$

where $z = ka$ and $z_0 = \sqrt{\frac{2mV_0}{\hbar^2}}a$.

(a) Show that the odd parity bound states can be found by solving the equation

$$-\cot z = +\sqrt{\left(\frac{z_0}{z}\right)^2 - 1}.$$

Now consider the *half*-finite square well with potential

$$V(x) = \begin{cases} \infty, & x < 0, \\ 0, & 0 \le x \le a, \\ V_0, & x > a. \end{cases}$$

(b) How are the bound states of this potential related to those of the finite square well of Part (a)?

(c) Are there bound states of this potential for any $V_0 > 0$? If not, what is the maximum value of V_0 for which there are no bound states?

5.4 Probability current.

(a) Show that the probability current density for a one-dimensional quantum particle with position-space wave function

$$\psi(x) = Ae^{ikx} + Be^{-ikx},$$

where k is real valued, is

$$j = \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right).$$

(b) Show that the probability current density for any particle with a real-valued position space wave function is zero.

5.5 Delta function potential.

Consider a one-dimensional quantum particle of mass m in the presence of a delta-function potential $V(x) = -\alpha\delta(x)$ where α has units of energy times length.

(a) By applying the appropriate boundary conditions at $x = 0$, show that for $\alpha > 0$ and $E < 0$ this potential admits a bound state of energy $E = -m\alpha^2/(2\hbar^2)$. Are there any other bound states?

(b) Now consider a scattering process. Seek solutions for $E > 0$ of the form $\psi(x) = Ae^{ikx} + Be^{-ikx}$ for $x < 0$ and $\psi(x) = Ce^{ikx}$ for $x > 0$. By applying the appropriate boundary conditions at $x = 0$, obtain the reflection and transmission coefficients R and T for this potential.