Physics 5645
Quantum Mechanics A
Problem Set VI

Due: Tuesday, Oct 31, 2017

6.1 Consider a one-dimensional quantum particle in the ground state of an infinite square well of width L. Suddenly, the well is expanded symmetrically to twice its size, leaving the wave function undisturbed. What is the probability to find the particle in the ground state of the new well?

6.2 In class we showed that the normalized position-space wave function for the nth energy eigenstate of a one-dimensional quantum Harmonic oscillator can be expressed as (see also Eq. 2.3.32 in Sakurai and Napolitano, pg. 92),

$$\psi_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \frac{1}{x_0^{n+1/2}} \left(x - x_0^2 \frac{d}{dx} \right)^n e^{-\frac{x^2}{2x_0^2}}$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. We also found

$$\psi_0(x) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{x_0}} e^{-\frac{x^2}{2x_0^2}} \quad \text{and} \quad \psi_1(x) = \frac{1}{\pi^{1/4}\sqrt{2}} \frac{1}{x_0^{3/2}} 2x e^{-\frac{x^2}{2x_0^2}}.$$

(a) Construct $\psi_2(x)$.

(b) Sketch ψ_0, ψ_1, and ψ_2.

(c) Verify that ψ_0, ψ_1, and ψ_2 are all orthogonal to one another by direct integration.

6.3 Compute the uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ for the nth energy eigenstate of a one-dimensional quantum harmonic oscillator and verify that the uncertainty principle is satisfied for all n.

6.4 Consider a one-dimensional quantum harmonic oscillator which, at time $t = 0$, is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|n\rangle + i|n+1\rangle \right).$$

(a) Working in the Schrödinger picture, find the time dependent state $|\psi(t)\rangle$ and compute the time-dependent expectation values $\langle\hat{x}\rangle = \langle\psi(t)|\hat{x}|\psi(t)\rangle$ and $\langle\hat{p}\rangle = \langle\psi(t)|\hat{p}|\psi(t)\rangle$.

(b) Working in the Heisenberg picture, again compute the time-dependent expectation values $\langle\hat{x}\rangle = \langle\psi(0)|\hat{x}(t)|\psi(0)\rangle$ and $\langle\hat{p}\rangle = \langle\psi(0)\hat{p}(t)|\psi(0)\rangle$ and verify that the result is the same as that obtained in Part (a).

6.5 Harmonic Oscillator Coherent States.

In Problem 6.2 you found that the ground state of the harmonic oscillator minimizes the uncertainty product with $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$, but for all of the excited states this product is greater than $\hbar^2/4$. There are, however, certain linear combinations of energy eigenstates, known as *coherent states*, for which the uncertainty product is minimized.

A coherent state is defined to be an eigenstate of the (non-Hermitian) lowering operator,

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

where α can be any complex number. (Since a is not Hermitian its eigenvalues are not required to be real.)

(a) Calculate $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{p}^2\rangle$ in the state $|\alpha\rangle$. In doing this do not assume α is real. Compute $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ and show that that the uncertainty product is $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$, and hence is minimized.

(b) Consider the expansion of $|\alpha\rangle$ in the energy basis,

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle.$$

Show that the expansion coefficients are $c_n = \frac{\alpha^n}{\sqrt{n!}}c_0$.

(c) By normalizing $|\alpha\rangle$ show that $c_0 = e^{-|\alpha|^2/2}$.

It follows from (b) and (c) that the probability to find the system in the state $|n\rangle$ is

$$P_n = |c_n|^2 = \frac{(|\alpha|^2)^n}{n!}e^{-|\alpha|^2},$$

which has the form of a Poisson distribution.

(d) Consider the limit of large quantum number n and, using Stirling's approximation, determine the most probable value of n (i.e., the value of n which maximizes P_n).

(e) Now consider the time evolution of this state in the Schrödinger picture. Show that the time-dependent state $|\alpha(t)\rangle$ remains an eigenstate of a but now with a time-dependent eigenvalue,

$$a|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle; \quad \alpha(t) = e^{-i\omega t}\alpha(0).$$

It follows that coherent states *stay* coherent states and so continue to have the minimal uncertainty product.

(f) Compute the commutator $[a, T(l)]$ where $T(l) = e^{-i\hat{p}l}$ is the translation operator. Using your result show that $T(l)|0\rangle$ (i.e. the ground state translated through distance l) is an eigenstate of a (and hence a coherent state), and determine the corresponding eigenvalue.

(g) (Optional: Inspired by the Oct. 26 colloquium — just for fun!) For a given $\alpha \neq 0$, consider the following *superposition* of coherent states,

$$|\psi_{qubit}\rangle = c_1\left(|\alpha\rangle + |-\alpha\rangle\right) + c_2\left(|i\alpha\rangle + |-i\alpha\rangle\right). \tag{1}$$

(This is the wave function for the "protected qubit" described in the colloquium if we interpret the n quantum number as the number of photons in a cavity. The "encoded" qubit is in the state $c_1|0\rangle + c_2|1\rangle$.)

Show that if you apply the lowering operator once to this state (corresponding to one photon escaping the cavity) the state changes. But if you apply the lowering operator *four* times (corresponding to *four* photons escaping the cavity) the state is unchanged.