Physics 5645
Quantum Mechanics A
Problem Set VII

Due: Tuesday, Nov 7, 2017

7.1 Consider a one-dimensional Harmonic oscillator with Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

The system is prepared in an initial state at time $t = 0$ with position-space wave function

$$\psi(x, 0) = Ax^2 e^{-\frac{m\omega}{2\hbar}x^2}.$$

(a) Determine the normalization constant A.

(b) Express the normalized $\psi(x, 0)$ as a linear combination of Harmonic oscillator eigenstates.

(c) Working in the Schrödinger picture, find the time-dependent position-space wave function $\psi(x, t)$.

(d) Show that the expectation value of $\hat{x}$ in this state is zero at all times. Find the expectation value of $\hat{x}^2$ in this state as a function of time and hence determine $\langle(\Delta x)^2\rangle$. Can you interpret your result?

7.2 Schrödinger equation in momentum-space representation.

(a) Using the fact that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar},$$

show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle.$$

Now consider a one-dimensional harmonic oscillator with Hamiltonian,

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

(b) Obtain the momentum-space representation of the time-independent Schrödinger equation, $H|E\rangle = E|E\rangle$. This will be a differential equation for momentum-space wave function $\tilde{\psi}_E(p) = \langle p|E\rangle$.

(c) Based on the form of this equation, determine the momentum-space wave functions for the energy eigenstates. [Note: there is no need to solve this equation "from scratch." You should be able to write down the eigenfunctions by analogy with the position-space Schrödinger equation.]

7.3 Half-harmonic potential.

A one-dimensional quantum particle of mass m is subject to the "half-harmonic" potential

$$V(x) = \begin{cases} \frac{1}{2}kx^2 & \text{for } x > 0, \\ \infty & \text{for } x < 0. \end{cases}$$

(a) What is the ground state energy for this particle?

(b) Determine the normalized position-space wave function for the ground state.

Now assume the particle is in the ground state when the potential suddenly changes to a *full* harmonic potential, $V(x) = \frac{1}{2}kx^2$ for $-\infty < x < \infty$.

(c) What is the probability to find the particle in the ground state of the new potential?

(d) What is the probability to find the particle in the first-excited state of the new potential?

7.4 Uncertainty estimates.

Estimate the zero-point energy for a particle of mass m in the following potentials

(a) $V(x) = \alpha x^4$ in one dimension where $\alpha > 0$.

(b) $V(r) = -e^2/r$ in three dimensions. This is the potential felt by an electron in a hydrogen atom. Express the estimate in eV, taking e and m to be the charge and mass of the electron.

7.5 Particle in a three-dimensional box.

Consider a three-dimensional quantum particle of mass m confined to cubic box of volume L^3. Choose the origin of your coordinate system to be one of the corners of the cube so that the potential is 0 in the region $0 < x < L$, $0 < y < L$, $0 < z < L$, and infinite everywhere else.

(a) Obtain the normalized energy eigenfunctions and corresponding energy eigenvalues for this particle.

(b) Determine the degeneracies of the first six energy eigenvalues.

7.6 Three-dimensional harmonic oscillator.

A three-dimensional quantum particle of mass m experiences the harmonic potential

$$V(x) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2).$$

(a) Show that the (in general degenerate) energy eigenvalues are

$$E_n = \left(n + \frac{3}{2}\right) \hbar\omega, \quad n = 0, 1, 2, 3, \cdots$$

(b) Write down the corresponding position-space eigenfunctions for this particle in terms of the one-dimensional Harmonic oscillator wave functions. Reexpress the first four states in spherical coordinates.

(c) Show that the degeneracy of the energy level $E_n = (n + 3/2)\hbar\omega$ is $(n + 1)(n + 2)/2$.