Physics 5645
Quantum Mechanics A
Problem Set VIII

Due: Tuesday, Nov 14, 2017

8.1 Infinitesimal Rotations, Part I.

Let $R(\hat{n}\phi)$ by the rotation matrix which determines how the components of a vector $\vec{v}$ transform under rotation through angle ϕ about axis $\hat{n}$. For rotations about the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ axes these matrices are,

$$R(\hat{\imath}\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \; R(\hat{\jmath}\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \; R(\hat{k}\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Verify that, for $|\epsilon_x|, |\epsilon_y| \ll 1$,

$$R(-\epsilon_y\hat{\jmath})R(-\epsilon_x\hat{\imath})R(\epsilon_y\hat{\jmath})R(\epsilon_x\hat{\imath}) = R(-\epsilon_x\epsilon_y\hat{k}) + \cdots,$$

where $\cdots$ corresponds to terms which are of order ϵ_x^2, ϵ_y^2 or higher.

Hint: To do this it is enough to expand the relevant R matrices to *first* order in ϵ_x and ϵ_y. Then, when multiplying these matrices out, you can drop any terms of order ϵ_x^2, ϵ_y^2 or higher.

8.2 Infinitesimal Rotations, Part II.

Let $D(R(\hat{n}\phi))$ be the unitary operator which rotates quantum states about the axis $\hat{n}$ through the angle ϕ. For an infinitesimal rotation we have

$$D(R(\hat{n}d\phi)) = 1 - i\frac{\hat{n}\cdot\vec{J}}{\hbar}d\phi.$$

where $\vec{J} = (J_x, J_y, J_z)$ is the angular momentum operator.

Verify that

$$D(R(-\epsilon_y\hat{\jmath}))D(R(-\epsilon_x\hat{\imath}))D(R(\epsilon_y\hat{\jmath}))D(R(\epsilon_x\hat{\imath})) = 1 + \frac{1}{\hbar^2}[J_x, J_y]\epsilon_x\epsilon_y + \cdots,$$

where, $\vec{J}$ is the vector angular momentum operator, and, again, $\cdots$ indicates terms which are of order ϵ_x^2 or ϵ_y^2 or higher.

By comparing your result to that of Problem 8.1, deduce the fundamental angular momentum commutation relation,

$$[J_x, J_y] = i\hbar J_z.$$

8.3 Prove the identity

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b}\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\sigma,$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\mathbb{1}$ is the 2×2 identity matrix in the following two different ways.

(a) By expressing the product $\sigma_i\sigma_j$ in terms of the commutator $[\sigma_i, \sigma_j]$ and the anticommutator $\{\sigma_i, \sigma_j\}$ and using the fact that

$$[\sigma_i, \sigma_j] = 2i\sum_k \epsilon_{ijk}\sigma_k,$$

and

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}.$$

(b) By using the fact that any 2×2 matrix M can be expressed as

$$M = c\mathbb{1} + \vec{d}\cdot\sigma,$$

where

$$c = \frac{1}{2}Tr[M] \text{ and } d_i = \frac{1}{2}Tr[\sigma_i M],$$

and also using

$$Tr[\sigma_i\sigma_j] = 2\delta_{ij}\mathbb{1}, \quad Tr[\sigma_i\sigma_j\sigma_k] = 2i\epsilon_{ijk}\mathbb{1}.$$

8.4 Compute the following product of spin-1/2 rotation operators which correspond to a $\pi/2$ rotation about the x axis followed by a $\pi/2$ rotation about the y axis,

$$D(R(\hat{j}(\pi/2))D(R(\hat{i}(\pi/2)) = e^{-i\frac{S_y}{\hbar}\frac{\pi}{2}}e^{-i\frac{S_x}{\hbar}\frac{\pi}{2}}.$$

Express your answer in the form $\cos\frac{\phi}{2}\mathbb{1} - i\hat{n}\cdot\sigma\sin\frac{\phi}{2}$ and determine ϕ and $\hat{n}$, i.e. determine the rotation angle and axis of the net rotation produced by these two successive rotations.

Repeat your calculation for the case that the rotations are carried out in the opposite order

$$D(R(\hat{i}(\pi/2))D(R(\hat{j}(\pi/2)) = e^{-i\frac{S_x}{\hbar}\frac{\pi}{2}}e^{-i\frac{S_y}{\hbar}\frac{\pi}{2}}.$$

8.5 Let $|\psi\rangle_R$ be the state of a spin-1/2 particle obtained by applying the rotation operator for a z-axis rotation through angle ϕ to the state $|\psi\rangle$,

$$|\psi\rangle_R = e^{-iS_z\phi/\hbar}|\psi\rangle.$$

In class we showed that

$$_R\langle\psi|S_x|\psi\rangle_R = \cos\phi\langle\psi|S_x|\psi\rangle - \sin\phi\langle\psi|S_y|\psi\rangle.$$

Show that

$$_R\langle\psi|S_y|\psi\rangle_R = \sin\phi\langle\psi|S_x|\psi\rangle + \cos\phi\langle\psi|S_y|\psi\rangle,$$

and

$$_R\langle\psi|S_z|\psi\rangle_R = \langle\psi|S_z|\psi\rangle.$$

That is, show that

$$\begin{pmatrix} _R\langle\psi|S_x|\psi\rangle_R \\ _R\langle\psi|S_y|\psi\rangle_R \\ _R\langle\psi|S_z|\psi\rangle_R \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \langle\psi|S_x|\psi\rangle \\ \langle\psi|S_y|\psi\rangle \\ \langle\psi|S_z|\psi\rangle \end{pmatrix},$$

so that $\langle\vec{S}\rangle$ transforms under rotations as a vector (see $R(\hat{k}\phi)$ in Problem 8.1).