Physics 5645
Quantum Mechanics A
Problem Set IX

Due: Thursday, Nov 30, 2017

9.1 Constructing spherical harmonics.

(a) Use the fact that $L_z|11\rangle = \hbar|11\rangle$ and $L_+|11\rangle = 0$ and the position representations of L_+ and L_z,

$$\begin{aligned} \langle \vec{r}|L_+|\psi\rangle &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\langle\vec{r}|\psi\rangle, \\ \langle\vec{r}|L_z|\psi\rangle &= \frac{\hbar}{i}\frac{\partial}{\partial\phi}\langle\vec{r}|\psi\rangle, \end{aligned}$$

to explicitly derive $Y_1^1(\theta,\phi)$. Normalize it by carrying out the appropriate spherical integral. To be consistent with the usual convention be sure to include the appropriate factor of $(-1)^l$.

(b) By repeatedly applying the lowering operator L_- using the property that $L_-|l,m\rangle = \hbar\sqrt{(l+m)(l-m+1)}|l,m-1\rangle$ and the position representation of L_-,

$$\langle\vec{r}|L_-|\psi\rangle = -\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\langle\vec{r}|\psi\rangle,$$

obtain $Y_1^0(\theta,\phi)$ and $Y_1^{-1}(\theta,\phi)$. Compare your results with the familiar expressions,

$$Y_1^0(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta, \qquad Y_1^{\pm1}(\theta,\phi) = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}.$$

9.2 Parity and angular momentum.

Prove that under the parity operation $\vec{r}\to-\vec{r}$,

$$Y_l^m \to (-1)^l Y_l^m.$$

Hint: Show that this is true for Y_l^l, for which you have a simple explicit form, and then verify that applying L_- does not alter the parity.

9.3 Spin-1 rotation matrix.

Recall that the matrix representation of J_x for a spin-1 particle is,

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Compute the matrices ${J_x^{(1)}}^2$ and ${J_x^{(1)}}^3$. You should find that ${J_x^{(1)}}^3 = \hbar^2 J_x^{(1)}$.

(b) By Taylor expanding the exponential, and using the results of Part (a), show that the spin-1 matrix representation of the rotation operator for an x-axis rotation by angle ϕ, $D(R(\phi\hat{i})) = e^{-i\frac{J_x}{\hbar}\phi}$, is

$$e^{-i\frac{J_x^{(1)}}{\hbar}\phi} = \mathbb{1} - i\sin\phi\left(\frac{J_x^{(1)}}{\hbar}\right) + (\cos\phi - 1)\left(\frac{J_x^{(1)}}{\hbar}\right)^2.$$

9.4 Spherical harmonics in Cartesian coordinates.

(a) Show that the $l = 1$ spherical harmonics (see 9.1) can be expressed in Cartesian coordinates as,

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\frac{z}{r}, \qquad Y_1^{\pm 1} = \mp\left(\frac{3}{4\pi}\right)^{1/2}\frac{x \pm iy}{2^{1/2}r}.$$

(b) Now consider a particle in a state for which the position-space wave function is,

$$\psi(\vec{r}) = A(x + y + 2iz)e^{-\alpha r},$$

where A is a normalization constant. Using the result of Part (a), determine the probabilities for all possible results of an L_z measurement on this particle.

9.5 Rotating Y_l^ms.

Under a rotation through angle ϕ about the x-axis the x, y, and z components of the position vector of a particle transform as

$$\begin{aligned} x &\to x \\ y &\to y\cos\phi - z\sin\phi \\ z &\to z\cos\phi + y\sin\phi \end{aligned}$$

Thus, under this rotation, the spherical harmonic Y_1^0 must transform as

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \frac{z}{r} \to \left(\frac{3}{4\pi}\right)^{1/2} \frac{z\cos\phi - y\sin\phi}{r}$$

(The difference in sign of the $y\sin\phi$ term is due to the fact that we are consider an *active* rotation.)

(a) Expand the rotated Y_1^0 in terms of the unrotated Y_1^1, Y_1^0, and Y_1^{-1}.

(b) Compare your result with that obtained using the spin-1 matrix representation of the x-axis rotation operator found in 9.3. **Hint:** Apply the rotation matrix to the column vector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

which corresponds to the state Y_1^0.

(c) Repeat Parts (a) and (b) for the rotated Y_1^1 and Y_1^{-1}.

9.6 Spin-3/2 matrix representation of J_x.

(a) Using

$$J_\pm|j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle, \quad \text{where} \quad J_\pm = J_x \pm iJ_y,$$

determine $J_x^{(3/2)}$, the 4×4 matrix representation of J_x for $j = 3/2$ in the J_z basis.

(b) Find the eigenvalues of $J_x^{(3/2)}$ and confirm that they are $-\frac{3}{2}\hbar$, $-\frac{1}{2}\hbar$, $\frac{1}{2}\hbar$, $\frac{3}{2}\hbar$.