Physics 5645
Quantum Mechanics A
Problem Set X

Due: Friday, Dec 7, 2018

10.1 Commutation relations.

Using the fundamental commutation relations $[\hat{r}_i, \hat{r}_j] = [\hat{p}_i, \hat{p}_j] = 0$, and $[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}$, show that $[\vec{L}, \hat{\vec{r}}^2] = 0$ and $[\vec{L}, \hat{\vec{p}}^2] = 0$, where $\hat{\vec{r}}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$ and $\hat{\vec{p}}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$.

10.2 Finite spherical well.

A three-dimensional quantum particle of mass m is confined by the potential

$$V(r) = \begin{cases} -V_0 & r < a, \\ 0 & r \geq a \end{cases}$$

where $V_0 > 0$.

(a) Show that the $l = 0$ bound states occur when,

$$ka \cot ka = -\rho a,$$

where $k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$ and $\rho = \sqrt{\frac{-2mE}{\hbar^2}}$. (Note the similarity of this problem to Problem 6.2.)

This potential provides a crude approximation to the potential energy of the deuteron (proton-neutron bound state) as a function of proton-neutron separation, r. In what follows take m to be the proton-neutron reduced mass ($m = m_p m_n/(m_p + m_n) \simeq 470$ MeV/c^2), a to be the approximate size of the deuteron measured from scattering experiments, $a = 1.5$ fm, and use the fact that the binding energy of the deuteron, determined from mass measurements, is $W = 2.23$ Mev.

(b) Determine the value of V_0 in MeV.

(c) Determine whether or not the deuteron has any excited but still bound $l = 0$ states.

10.3 Three-dimensional isotropic harmonic oscillator.

Consider a quantum particle of mass m moving in the presence of a three-dimensional harmonic potential $V(r) = \frac{1}{2}m\omega^2 r^2$. Since the potential is spherically symmetric, we know that energy eigenstates can be taken to be simultaneous eigenstates of $\vec{L}^2$ and L_z. The position-space wave functions of these eigenstates will then have the form $\psi(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi)$.

(a) Write down the radial equation for the function $u(r) = rR(r)$.

(b) Introduce the dimensionless radial coordinate $\rho = \sqrt{\frac{m\omega}{\hbar}}r$ and let $\epsilon = \frac{2E}{\hbar\omega}$ where E is the energy of the state and show that the radial equation can be written,

$$\left(-\frac{d^2}{d\rho^2} + \rho^2 + \frac{l(l+1)}{\rho^2}\right)u(\rho) = \epsilon u(\rho). \tag{1}$$

It can be shown that the solutions to (1) have the form $u(\rho) = e^{-\rho^2/2}v(\rho)$ where

$$v(\rho) = \rho^{l+1}(a_0 + a_2\rho^2 + a_4\rho^4 + \cdots) = \rho^{l+1}\sum_{q=0,2,4,\cdots}^{\infty} a_q\rho^q.$$

(Feel free to show this yourself, but for purposes of this problem it is OK to just assume it.)

(c) Show that if $u(\rho) = e^{-\rho^2/2}v(\rho)$ the radial equation (1) implies that $v(\rho)$ satisfies the following equation,

$$\frac{d^2v}{d\rho^2} - 2\rho\frac{dv}{d\rho} + (\epsilon - 1)v - \frac{l(l+1)}{\rho^2}v = 0. \tag{2}$$

(d) Plug in the power series expression for $v(\rho)$ into (2) and obtain a recursion relation for the coefficients a_q. Determine the quantized values of E for which the series truncates.

(e) Construct the normalized ground state wave function (with $l = 0$ and $E = \frac{3}{2}\hbar\omega$) and the three first excited state wave functions (with $l = 1$, $m = -1, 0, 1$ and $E = \frac{5}{2}\hbar\omega$) for this particle.

10.4 Consider a He^+ ion which consists of a single electron orbiting a nucleus of charge $+2e$. If the nucleus of this atom absorbs a positron the nuclear charge will suddenly become $+3e$ (i.e. the ion will become a Li^{2+} ion). Assume the electron in the Helium ion was in its ground state before the positron absorption.

(a) What is the probability that, immediately after the positron absorption, the electron will be found in the ground state ($n = 1$) of the Li^{2+} ion?

(b) What is the probability that, immediately after the positron absorption, the electron will be found in each of the four degenerate $n = 2$ excited states of the Li^{2+} ion?

You may use the sudden approximation which assumes that the system remains in the ground state of the Hydrogen-like ion immediately after absorbing the positron.

10.5 Virial Theorem.

Consider a three-dimensional quantum particle with Hamiltonian,

$$H = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{r}}).$$

(a) Obtain the Heisenberg equation of motion for the operator $\Omega = \hat{\vec{r}} \cdot \hat{\vec{p}}$.

(b) Use your result from Part (a) and the fact that in a stationary state $\langle \Omega \rangle$ is time-independent to show that if $|\psi\rangle$ is an eigenstate of H then

$$\langle \psi | T | \psi \rangle = \frac{1}{2} \langle \psi | \hat{\vec{r}} \cdot \vec{\nabla} V(\hat{\vec{r}}) | \psi \rangle,$$

where

$$T = \frac{\hat{\vec{p}}^2}{2m}$$

is the kinetic energy operator. This is the quantum-mechanical version of the virial theorem.

(c) Apply your result from Part (b) to the case of a spherically symmetric potential of the form with $V(r) = kr^{\alpha}$ and show that in this case the virial theorem states that,

$$\langle T \rangle = \frac{\alpha}{2} \langle V \rangle, \tag{3}$$

in any energy eigenstate.

(d) Verify that (3) holds for the ground state of the isotropic harmonic oscillator ($\alpha = 2$) and the ground state of the hydrogen atom ($\alpha = -1$).