4.1 Show that if the position-space wave function for a particle, \( \psi(x) = \langle x|\psi \rangle \), is real valued then the expectation value of the momentum operator in that state is \( \langle \hat{p} \rangle = 0 \). (Hint: Show that the probabilities to measure momenta of \( +p \) and \( -p \) are equal.) Also show that multiplying \( \psi(x) \) by a constant, even if complex, does not change this result. (It better not, since the states \( |\psi\rangle \) and \( c|\psi\rangle \) are physically equivalent.)

4.2 Problem 1.27, Sakurai and Napolitano, Pg. 63.

4.3 Problem 1.32 Part (b), Sakurai and Napolitano, Pg. 65.

4.4 A one-dimensional particle is in the state \( |\psi\rangle = \int \psi(x)|x\rangle dx \) where \( \psi(x) = \langle x|\psi \rangle \), the position-space wave function, is given by

\[
\psi(x) = \begin{cases} 
A & -a \leq x \leq a, \\
0 & \text{otherwise}.
\end{cases}
\]

(a) Find \( A \) so that \( |\psi\rangle \) is normalized and compute the momentum-space wave function for this state, \( \tilde{\psi}(p) = \langle p|\psi \rangle \).

(b) If an experiment were performed to measure the momentum \( p \) of the particle in this state, what is the probability that the result would be such that \( |p| \geq \hbar/a \)? [To answer this you may need to do an integral numerically.]

4.5 Problem 2.2, Sakurai and Napolitano, Pg. 148.

4.6 Consider the precession of the spin of an electron in a uniform magnetic field \( \vec{B} = B\hat{k} \) for which the Hamiltonian is

\[
H = -\frac{eB}{mc}S_z = \omega S_z,
\]

where \( \omega = \frac{|e|B}{mc} \).
At time $t = 0$ the electron spin is in the state,

$$|\psi(0)\rangle = \cos \frac{\theta}{2} |+\rangle + \sin \frac{\theta}{2} |-\rangle,$$

(i.e., an eigenstate of the operator $\hat{n} \cdot \mathbf{S}$ with eigenvalue $+\hbar/2$ where $\hat{n}$ is in the $xz$ plane and makes an angle $\theta$ with the $z$ axis.) In what follows work within the Schrödinger picture.

(a) Obtain the time dependent state $|\psi(t)\rangle$.

(b) Find the probability for finding the electron in the $S_x = \hbar/2$ state as a function of time.

(c) Obtain the expectation values of $S_x$, $S_y$, and $S_z$ as a function of time. Describe the time evolution of the vector $\langle \mathbf{S} \rangle = \langle \psi(t)|\mathbf{S}|\psi(t)\rangle$.

4.7 Time Evolution of free particle wave packet (Schrödinger picture).

Consider the motion of a free particle in one-dimension with Hamiltonian

$$H = \frac{\hat{p}^2}{2m}.$$

At time $t = 0$ the particle is in a state $|\psi(0)\rangle$ with position-space wave function

$$\langle x|\psi(0)\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp^{-x^2/(2d^2)} e^{ikx}.$$

(a) Working in the Schrödinger picture, find the position space wave function at time $t$.

Answer:

$$\langle x|\psi(t)\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \frac{\exp^{-(x-v_gt)^2/[2d^2(1+i\hbar/m_d^2t)]}}{\sqrt{1 + i\hbar/m_d^2t}} e^{i(k-x-v_pt)}, \quad (1)$$

where $v_g = \hbar k/m$ is the group velocity, and $v_p = \hbar k/(2m)$ is the phase velocity.

(b) Specializing to the case $k = 0$, compute the expectation values $\langle \hat{x} \rangle$, $\langle \hat{x}^2 \rangle$, $\langle \hat{p} \rangle$, and $\langle \hat{p}^2 \rangle$. Determine $\langle (\Delta x)^2 \rangle$ and $\langle (\Delta p)^2 \rangle$ as a function of time and verify that the uncertainty principle always holds.