8.1 Uncertainty estimates.

Estimate the zero-point energy for a particle of mass $m$ in the following potentials

(a) $V(x) = \alpha x^4$ in one dimension where $\alpha > 0$.

(b) $V(r) = -e^2/r$ in three dimensions. This is the potential felt by an electron in a hydrogen atom. Express the estimate in eV, taking $e$ and $m$ to be the charge and mass of the electron.

8.2 Particle in a three-dimensional box.

Consider a three-dimensional quantum particle of mass $m$ confined to cubic box of volume $L^3$. Choose the origin of your coordinate system to be one of the corners of the cube so that the potential is 0 in the region $0 < x < L$, $0 < y < L$, $0 < z < L$, and infinite everywhere else.

(a) Obtain the normalized energy eigenfunctions and corresponding energy eigenvalues for this particle.

(b) Determine the degeneracies of the first six energy eigenvalues.

8.3 Three-dimensional harmonic oscillator.

A three-dimensional quantum particle of mass $m$ experiences the harmonic potential

$$V(x) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2).$$

(a) Show that the (in general degenerate) energy eigenvalues are

$$E_n = \left(n + \frac{3}{2}\right)\hbar\omega, \quad n = 0, 1, 2, 3, \cdots$$

(b) Write down the corresponding position-space eigenfunctions for this particle in terms of the one-dimensional Harmonic oscillator wave functions. Reexpress the first four states in spherical coordinates.
(c) Show that the degeneracy of the energy level \( E_n = (n + 3/2)\hbar \omega \) is \((n + 1)(n + 2)/2\).

8.4 Infinitesimal Rotations.

Let \( R(\hat{n} \phi) \) by the rotation matrix which determines how the components of a vector \( \vec{v} \) transform under rotation through angle \( \phi \) about axis \( \hat{n} \). For rotations about the \( \hat{i} \), \( \hat{j} \), and \( \hat{k} \) axes these matrices are,

\[
R(\hat{i} \phi) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi - \sin \phi & 0 \\
0 & \sin \phi & \cos \phi
\end{pmatrix},
\]

\[
R(\hat{j} \phi) = \begin{pmatrix}
\cos \phi & 0 & \sin \phi \\
0 & 1 & 0 \\
-\sin \phi & 0 & \cos \phi
\end{pmatrix},
\]

\[
R(\hat{k} \phi) = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(a) Verify that, for \( |\epsilon_x|, |\epsilon_y| \ll 1 \),

\[
R(-\epsilon_y \hat{j})R(-\epsilon_x \hat{i})R(\epsilon_y \hat{j})R(\epsilon_x \hat{i}) = R(-\epsilon_x \epsilon_y \hat{k}) + \cdots,
\]

where \( \cdots \) corresponds to terms which are of order \( \epsilon_x^2, \epsilon_y^2 \) or higher.

**Hint:** To do this it is enough to expand the relevant \( R \) matrices to first order in \( \epsilon_x \) and \( \epsilon_y \). Then, when multiplying these matrices out, you can drop any terms of order \( \epsilon_x^2, \epsilon_y^2 \) or higher.

Let \( D(R(\hat{n} \phi)) \) be the unitary operator which rotates quantum states about the axis \( \hat{n} \) through the angle \( \phi \). For an infinitesimal rotation we have

\[
D(R(\hat{n}d\phi)) = 1 - i \frac{\hat{n} \cdot \vec{J}}{\hbar} d\phi.
\]

where \( \vec{J} = (J_x, J_y, J_z) \) is the angular momentum operator.

(a) Verify that

\[
D(R(-\epsilon_y \hat{j}))D(R(-\epsilon_x \hat{i}))D(R(\epsilon_y \hat{j}))D(R(\epsilon_x \hat{i})) = 1 + \frac{1}{\hbar^2}[J_x, J_y] \epsilon_x \epsilon_y + \cdots,
\]

where, \( \vec{J} \) is the vector angular momentum operator, and, again, \( \cdots \) indicates terms which are of order \( \epsilon_x^2 \) or \( \epsilon_y^2 \) or higher.

By comparing your result to that of Part (a) above, deduce the fundamental angular momentum commutation relation,

\[
[J_x, J_y] = i\hbar J_z.
\]
8.5 Prove the identity
\[(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b} \mathbb{1} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma},\]
where \(\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)\) and \(\mathbb{1}\) is the 2x2 identity matrix in the following two different ways.

(a) By using the fact that
\[\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \sum_k \epsilon_{ijk} \sigma_k,\]
which you proved in Problem 3.2(a).

(b) By using the fact that any 2 \times 2 matrix \(M\) can be expressed as
\[M = c \mathbb{1} + \vec{d} \cdot \vec{\sigma},\]
where
\[c = \frac{1}{2} Tr[M] \quad \text{and} \quad d_i = \frac{1}{2} Tr[\sigma_i M],\]
which you proved in Problem 3.4(c), and also using
\[Tr[\sigma_i \sigma_j] = 2 \delta_{ij} \mathbb{1}, \quad Tr[\sigma_i \sigma_j \sigma_k] = 2i \epsilon_{ijk} \mathbb{1}.

8.6 Let \(|\psi\rangle_R\) be the state of a spin-1/2 particle obtained by applying the rotation operator for a z-axis rotation through angle \(\phi\) to the state \(|\psi\rangle\),
\[|\psi\rangle_R = e^{-iS_z \phi/h}|\psi\rangle.\]

In class we showed that
\[R\langle \psi | S_z | \psi \rangle_R = \cos \phi \langle \psi | S_x | \psi \rangle - \sin \phi \langle \psi | S_y | \psi \rangle.\]

Show that
\[R\langle \psi | S_y | \psi \rangle_R = \sin \phi \langle \psi | S_x | \psi \rangle + \cos \phi \langle \psi | S_y | \psi \rangle,\]
and
\[R\langle \psi | S_z | \psi \rangle_R = \langle \psi | S_z | \psi \rangle.\]
That is, show that

\[
\begin{pmatrix}
R(\langle \psi | S_x | \psi \rangle_R) \\
R(\langle \psi | S_y | \psi \rangle_R) \\
R(\langle \psi | S_z | \psi \rangle_R)
\end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
\langle \psi | S_x | \psi \rangle \\
\langle \psi | S_y | \psi \rangle \\
\langle \psi | S_z | \psi \rangle
\end{pmatrix},
\]

so that \(\langle \vec{S} \rangle\) transforms under rotations as a vector (see \(R(\hat{k}\phi)\) in Problem 8.4).