Physics 5645
Quantum Mechanics A
Problem Set IX

Due: Friday, Dec 6, 2019

10.1 Constructing spherical harmonics.

(a) Use the fact that $L_z|11\rangle = \hbar|11\rangle$ and $L_+|11\rangle = 0$ and the position representations of L_+ and L_z,

$$\begin{aligned}
\langle\vec{r}|L_+|\psi\rangle &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)\langle\vec{r}|\psi\rangle, \\
\langle\vec{r}|L_z|\psi\rangle &= \frac{\hbar}{i}\frac{\partial}{\partial\phi}\langle\vec{r}|\psi\rangle,
\end{aligned}$$

to explicitly derive $Y_1^1(\theta,\phi)$. Normalize it by carrying out the appropriate spherical integral. To be consistent with the usual convention be sure to include the appropriate factor of $(-1)^l$.

(b) By repeatedly applying the lowering operator L_- using the property that $L_-|l,m\rangle = \hbar\sqrt{(l+m)(l-m+1)}|l,m-1\rangle$ and the position representation of L_-,

$$\langle\vec{r}|L_-|\psi\rangle = -\hbar e^{-i\phi}\left(\frac{\partial}{\partial\theta} - i\cot\theta\frac{\partial}{\partial\phi}\right)\langle\vec{r}|\psi\rangle,$$

obtain $Y_1^0(\theta,\phi)$ and $Y_1^{-1}(\theta,\phi)$. Compare your results with the familiar expressions,

$$Y_1^0(\theta,\phi) = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta, \qquad Y_1^{\pm1}(\theta,\phi) = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}.$$

10.2 Parity and angular momentum.

Prove that under the parity operation $\vec{r}\to-\vec{r}$,

$$Y_l^m \to (-1)^l Y_l^m.$$

Hint: Show that this is true for Y_l^l, for which you have a simple explicit form, and then verify that applying L_- does not alter the parity.

10.3 Spherical harmonics in Cartesian coordinates.

(a) Show that the $l = 1$ spherical harmonics (see 10.1) can be expressed in Cartesian coordinates as,

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \frac{z}{r}, \qquad Y_1^{\pm 1} = \mp \left(\frac{3}{4\pi}\right)^{1/2} \frac{x \pm iy}{2^{1/2} r}.$$

(b) Now consider a particle in a state for which the position-space wave function is,

$$\psi(\vec{r}) = A(x + y + 2iz)e^{-\alpha r},$$

where A is a normalization constant. Using the result of Part (a), determine the probabilities for all possible results of an L_z measurement on this particle.

10.4 Rotating Y_l^ms.

Under a rotation through angle ϕ about the x-axis the x, y, and z components of the position vector of a particle transform as

$$\begin{aligned} x &\rightarrow x \\ y &\rightarrow y\cos\phi - z\sin\phi \\ z &\rightarrow z\cos\phi + y\sin\phi \end{aligned}$$

Thus, under this rotation, the spherical harmonic Y_1^0 must transform as

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \frac{z}{r} \rightarrow \left(\frac{3}{4\pi}\right)^{1/2} \frac{z\cos\phi - y\sin\phi}{r}$$

(The difference in sign of the $y\sin\phi$ term is due to the fact that we are considering an *active* rotation.)

(a) Expand the rotated Y_1^0 in terms of the unrotated Y_1^1, Y_1^0, and Y_1^{-1}.

(b) Compare your result with that obtained using the spin-1 matrix representation of the x-axis rotation operator found in Problem 9.3. **Hint:** Apply the rotation matrix to the column vector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

which corresponds to the state Y_1^0.

(c) Repeat Parts (a) and (b) for the rotated Y_1^1 and Y_1^{-1}.

10.5 Commutation relations.

Using the fundamental commutation relations $[\hat{r}_i, \hat{r}_j] = [\hat{p}_i, \hat{p}_j] = 0$, and $[\hat{r}_i, \hat{p}_j] = i\hbar\delta_{ij}$, show that $[\vec{L}, \hat{\vec{r}^2}] = 0$ and $[\vec{L}, \hat{\vec{p}^2}] = 0$, where $\hat{\vec{r}^2} = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$ and $\hat{\vec{p}^2} = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2$.

10.6 Finite spherical well.

A three-dimensional quantum particle of mass m is confined by the potential

$$V(r) = \begin{cases} -V_0 & r < a, \\ 0 & r \geq a \end{cases}$$

where $V_0 > 0$.

(a) Show that the $l = 0$ bound states occur when,

$$ka \cot ka = -\rho a,$$

where $k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$ and $\rho = \sqrt{\frac{-2mE}{\hbar^2}}$. (Note the similarity of this problem to Problem 6.2.)

This potential provides a crude approximation to the potential energy of the deuteron (proton-neutron bound state) as a function of proton-neutron separation, r. In what follows take m to be the proton-neutron reduced mass ($m = m_p m_n/(m_p + m_n) \simeq 470$ MeV/c^2), a to be the approximate size of the deuteron measured from scattering experiments, $a =$ 1.5 fm, and use the fact that the binding energy of the deuteron, determined from mass measurements, is $W = 2.23$ Mev.

(b) Determine the value of V_0 in MeV.

(c) Determine whether or not the deuteron has any excited but still bound $l = 0$ states.

10.7 Three-dimensional isotropic harmonic oscillator.

Consider a quantum particle of mass m moving in the presence of a three-dimensional harmonic potential $V(r) = \frac{1}{2}m\omega^2 r^2$. Since the potential is spherically symmetric, we know that energy eigenstates can be taken to be simultaneous eigenstates of $\vec{L}^2$ and L_z. The position-space wave functions of these eigenstates will then have the form $\psi(r,\theta,\phi) = R(r)Y_l^m(\theta,\phi)$.

(a) Write down the radial equation for the function $u(r) = rR(r)$.

(b) Introduce the dimensionless radial coordinate $\rho = \sqrt{\frac{m\omega}{\hbar}}r$ and let $\epsilon = \frac{2E}{\hbar\omega}$ where E is the energy of the state and show that the radial equation can be written,

$$\left(-\frac{d^2}{d\rho^2} + \rho^2 + \frac{l(l+1)}{\rho^2}\right)u(\rho) = \epsilon u(\rho). \tag{1}$$

It can be shown that the solutions to (1) have the form $u(\rho) = e^{-\rho^2/2}v(\rho)$ where

$$v(\rho) = \rho^{l+1}(a_0 + a_2\rho^2 + a_4\rho^4 + \cdots) = \rho^{l+1}\sum_{q=0,2,4,\cdots}^{\infty} a_q\rho^q.$$

(Feel free to show this yourself, but for purposes of this problem it is OK to just assume it.)

(c) Show that if $u(\rho) = e^{-\rho^2/2}v(\rho)$ the radial equation (1) implies that $v(\rho)$ satisfies the following equation,

$$\frac{d^2v}{d\rho^2} - 2\rho\frac{dv}{d\rho} + (\epsilon - 1)v - \frac{l(l+1)}{\rho^2}v = 0. \tag{2}$$

(d) Plug in the power series expression for $v(\rho)$ into (2) and obtain a recursion relation for the coefficients a_q. Determine the quantized values of E for which the series truncates.

(e) Construct the normalized ground state wave function (with $l = 0$ and $E = \frac{3}{2}\hbar\omega$) and the three first excited state wave functions (with $l = 1$, $m = -1, 0, 1$ and $E = \frac{5}{2}\hbar\omega$) for this particle.

10.8 Consider a He^+ ion which consists of a single electron orbiting a nucleus of charge $+2e$. If the nucleus of this atom absorbs a positron the nuclear charge will suddenly become $+3e$ (i.e. the ion will become a Li^{2+} ion). Assume the electron in the Helium ion was in its ground state before the positron absorption.

(a) What is the probability that, immediately after the positron absorption, the electron will be found in the ground state ($n = 1$) of the Li^{2+} ion?

(b) What is the probability that, immediately after the positron absorption, the electron will be found in each of the four degenerate $n = 2$ excited states of the Li^{2+} ion?

You may use the sudden approximation which assumes that the system remains in the ground state of the Hydrogen-like ion immediately after absorbing the positron.

10.9 Virial Theorem.

Consider a three-dimensional quantum particle with Hamiltonian,

$$H = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{r}}).$$

(a) Obtain the Heisenberg equation of motion for the operator $\Omega = \hat{\vec{r}} \cdot \hat{\vec{p}}$.

(b) Use your result from Part (a) and the fact that in a stationary state $\langle \Omega \rangle$ is time-independent to show that if $|\psi\rangle$ is an eigenstate of H then

$$\langle \psi | T | \psi \rangle = \frac{1}{2} \langle \psi | \hat{\vec{r}} \cdot \vec{\nabla} V(\hat{\vec{r}}) | \psi \rangle,$$

where

$$T = \frac{\hat{\vec{p}}^2}{2m}$$

is the kinetic energy operator. This is the quantum-mechanical version of the virial theorem.

(c) Apply your result from Part (b) to the case of a spherically symmetric potential of the form with $V(r) = kr^\alpha$ and show that in this case the virial theorem states that,

$$\langle T \rangle = \frac{\alpha}{2} \langle V \rangle, \tag{3}$$

in any energy eigenstate.

(d) Verify that (3) holds for the ground state of the isotropic harmonic oscillator ($\alpha = 2$) and the ground state of the hydrogen atom ($\alpha = -1$).