

**Physics 5645**  
**Quantum Mechanics A**  
**Problem Set III**

Due: Tuesday, Sep 24, 2019

3.1 In Problem 1.3 (Problem 1.8 in Sakurai and Napolitano) you showed that

$$[S_i, S_j] = i\epsilon_{ijk}\hbar S_k \quad \text{and} \quad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij},$$

or, since  $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ ,

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \quad \text{and} \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1},$$

where  $\sigma_i$  are the Pauli matrices [the index  $i$  runs over Cartesian components  $x$ ,  $y$ , and  $z$  and repeated indices are summed] and  $\mathbb{1}$  is the  $2\times 2$  identity matrix.

(a) Show that

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k. \tag{1}$$

(b) Show that (1) together with the facts that  $Tr[\mathbb{1}] = 2$  and  $Tr[\sigma_i] = 0$  implies

$$Tr[\sigma_i\sigma_j] = 2\delta_{ij}.$$

Now consider a general  $2\times 2$  matrix (not necessarily Hermitian or Unitary) of the form

$$A = a_0\mathbb{1} + \vec{a} \cdot \vec{\sigma}$$

where  $a_0$  and  $a_i$  ( $i = x, y, z$ ) are complex numbers.

(c) How are  $a_0$  and  $a_i$  related to  $Tr[A]$  and  $Tr[\sigma_i A]$ ?

(d) Obtain  $a_0$  and  $a_i$  in terms of the matrix elements of  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ .

[**Note:** This problem gives a useful method for showing things like

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} = \frac{H_{11} + H_{22}}{2}\mathbb{1} + \frac{H_{11} - H_{22}}{2}\sigma_z + H_{12}\sigma_x,$$

which was useful for doing Problem 1.11 from Sakurai and Napolitano on the last Problem Set.]

3.2 Problem 1.17, Sakurai and Napolitano, Pg. 61.

3.3 Problem 1.23, Sakurai and Napolitano, Pg. 63

3.4 As I'm sure you know, and we will soon see, the eigenkets of the Hamiltonian operator for a one-dimensional particle confined between two rigid walls, one at  $x = 0$  and the other at  $x = a$ , are  $|\psi_n\rangle = \int \psi_n(x)|x\rangle dx$  where

$$\psi_n(x) = \begin{cases} A \sin \frac{n\pi x}{a} & \text{for } 0 < x < a, \\ 0 & \text{otherwise.} \end{cases}$$

Here  $n = 1, 2, 3, \dots$  and  $A$  is a normalization constant.

Compute the  $x$ - $p$  uncertainty products  $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$  for the states  $|\psi_n\rangle$  and verify that they satisfies the uncertainty principle for all  $n$ .

3.5 Problem 1.32 Part (a), Sakurai and Napolitano, Pg. 61. [You will do Part (b) of this problem on the next Problem Set.]