Physics 5645
Quantum Mechanics A
Problem Set V

Due: Tuesday, Oct 8, 2019

5.1 Time Evolution of free particle wave packet (Schrödinger picture).

Consider the motion of a free particle in one-dimension with Hamiltonian

$$H = \frac{\hat{p}^2}{2m}.$$

At time $t = 0$ the particle is in a state $|\psi(0)\rangle$ with position-space wave function

$$\langle x|\psi(0)\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp^{-x^2/(2d^2)} e^{ikx}.$$

(a) Working in the Schrödinger picture, find the position space wave function at time t.

Answer:

$$\langle x|\psi(t)\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \frac{\exp^{-(x-v_g t)^2/\left[2d^2\left(1+i\frac{\hbar}{md^2}t\right)\right]}}{\sqrt{1+i\frac{\hbar}{md^2}t}} e^{ik(x-v_p t)}, \tag{1}$$

where $v_g = \hbar k/m$ is the group velocity, and $v_p = \hbar k/(2m)$ is the phase velocity.

(b) Specializing to the case $k = 0$, compute the expectation values $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{p}^2\rangle$. Determine $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ as a function of time and verify that the uncertainty principle always holds.

5.2 Useful commutator identities.

(a) Prove the following commutator identity (Eq. 1.6.50e in Sakurai and Napolitano),

$$[A, BC] = [A, B]C + B[A, C].$$

(b) Use this identity, and the fact that $[\hat{x}, \hat{p}_x] = i\hbar$, to prove that

$$[\hat{x}, \hat{p}_x^n] = i\hbar n \hat{p}_x^{n-1}.$$

(c) Using the result of (b), show that

$$[\hat{x}, g(\hat{p}_x)] = i\hbar \frac{dg}{dp_x},$$

for any function g that can be expressed as a power series in its argument. Then show that the three-dimensional generalization of this result is (Eq. 2.2.23a in Sakurai and Napolitano),

$$[\hat{r}_i, G(\hat{\mathbf{p}})] = i\hbar \frac{\partial G}{\partial p_i},$$

where, again, G is a function that can be expressed as a power series in its arguments.

(d) Similarly show that (Eq. 2.2.23b in Sakurai and Napolitano).

$$[\hat{p}_i, F(\hat{\mathbf{r}})] = -i\hbar \frac{\partial F}{\partial r_i},$$

where F is a function that can be expressed as a power series in its arguments.

5.3 Finite translation operator.

(a) Using the results obtained in the previous problem, evaluate the commutator,

$$[\hat{r}_i, T(\mathbf{l})],$$

where

$$T(\mathbf{l}) = e^{-i\hat{\mathbf{p}}\cdot\mathbf{l}/\hbar},$$

is the translation operator for finite spatial displacement.

(b) Using the result of (a), demonstrate how the position expectation value $\langle\hat{\mathbf{r}}\rangle$ changes under translation.

5.4 As in Problem 4.6 on the last problem set, consider the precession of the spin of an electron in a uniform magnetic field $\vec{B} = B\hat{k}$ for which the Hamiltonian is

$$H = -\frac{eB}{mc} S_z = \omega S_z,$$

where $\omega = \frac{|e|B}{mc}$.

At time $t = 0$ the electron spin is in the state,

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|+\rangle + \sin\frac{\theta}{2}|-\rangle,$$

(i.e., an eigenstate of the operator $\hat{\mathbf{n}} \cdot \mathbf{S}$ with eigenvalue $+\hbar/2$ where $\hat{\mathbf{n}}$ is in the xz plane and makes an angle θ with the z axis.) In what follows work within the Heisenberg picture.

(a) Write down the Heisenberg equations of motion for the time-dependent operators $S_x(t)$, $S_y(t)$, and $S_z(t)$. Solve these equations to obtain expressions for $S_x(t)$, $S_y(t)$, and $S_z(t)$. [Your expressions should be in terms of t and $S_x(0)$, $S_y(0)$, and $S_z(0)$.]

(b) Using the results of Part (a), obtain the expectation values of S_x, S_y, and S_z as a function of time. Describe the time evolution of the vector $\langle \mathbf{S} \rangle = \langle \psi(0)|\mathbf{S}(t)|\psi(0)\rangle$ and show that it is the same as that obtained in Part (c) of Problem 4.6 using the Schrödinger picture.

5.5 Time evolution of free particle wave packet (Heisenberg picture).

Consider once more our good friend the Gaussian wave packet with position-space wave function,

$$\langle x|\psi\rangle = \frac{1}{\pi^{1/4}\sqrt{d}} e^{-x^2/(2d^2)}. \tag{2}$$

(a) Prove that for this state $\hat{x}|\psi\rangle = i(\text{real quantity})\hat{p}|\psi\rangle$.

Now, working in the Heisenberg picture, consider the time evolution of this state for the free-particle Hamiltonian,

$$H = \frac{\hat{p}^2}{2m}.$$

Assume that at time $t = 0$ (used to define the Heisenberg picture) the particle is in the state $|\psi\rangle$ given in (2).

(b) Write down the Heisenberg equations of motion for $\hat{x}(t)$ and $\hat{p}(t)$. Solve them to obtain expressions for these operators in terms of $\hat{x}(0)$, $\hat{p}(0)$, and t.

(c) Obtain $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ as a function of time for this particle and verify that the uncertainty principle is always satisfied.

In doing Part (c) you may use the following expectation values,

$$\begin{aligned} \langle \psi(0)|\hat{x}(0)|\psi(0)\rangle &= 0, \qquad \langle \psi(0)|\hat{p}(0)|\psi(0)\rangle = 0, \\ \langle \psi(0)|\hat{x}^2(0)|\psi(0)\rangle &= d^2/2, \quad \langle \psi(0)|\hat{p}^2(0)|\psi(0)\rangle = \hbar^2/(2d^2), \end{aligned}$$

since you've calculated them before. If done properly the calculation for Part (c) should not be lengthy. You may find the result of Part (a) useful.