Physics 5645
Quantum Mechanics A
Problem Set VII

Due: Thursday, Oct 31, 2019

7.1 Compute the uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ for the nth energy eigenstate of a one-dimensional quantum harmonic oscillator and verify that the uncertainty principle is satisfied for all n.

7.2 Consider a one-dimensional quantum harmonic oscillator which, at time $t = 0$, is in the state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|n\rangle + i|n+1\rangle\right).$$

(a) Working in the Schrödinger picture, find the time dependent state $|\psi(t)\rangle$ and compute the time-dependent expectation values $\langle\hat{x}\rangle = \langle\psi(t)|\hat{x}|\psi(t)\rangle$ and $\langle\hat{p}\rangle = \langle\psi(t)|\hat{p}|\psi(t)\rangle$.

(b) Working in the Heisenberg picture, again compute the time-dependent expectation values $\langle\hat{x}\rangle = \langle\psi(0)|\hat{x}(t)|\psi(0)\rangle$ and $\langle\hat{p}\rangle = \langle\psi(0)\hat{p}(t)|\psi(0)\rangle$ and verify that the result is the same as that obtained in Part (a).

7.3 Harmonic Oscillator Coherent States.

In Problem 7.1 you found that the ground state of the harmonic oscillator minimizes the uncertainty product with $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$, but for all of the excited states this product is greater than $\hbar^2/4$. There are, however, certain linear combinations of energy eigenstates, known as *coherent states*, for which the uncertainty product is minimized.

A coherent state is defined to be an eigenstate of the (non-Hermitian) lowering operator,

$$a|\alpha\rangle = \alpha|\alpha\rangle,$$

where α can be any complex number. (Since a is not Hermitian its eigenvalues are not required to be real.)

(a) Calculate $\langle\hat{x}\rangle$, $\langle\hat{x}^2\rangle$, $\langle\hat{p}\rangle$, and $\langle\hat{p}^2\rangle$ in the state $|\alpha\rangle$. In doing this do not assume α is real. Compute $\langle(\Delta x)^2\rangle$ and $\langle(\Delta p)^2\rangle$ and show that that the uncertainty product is $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle = \hbar^2/4$, and hence is minimized.

(b) Consider the expansion of $|\alpha\rangle$ in the energy basis,

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle.$$

Show that the expansion coefficients are $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$.

(c) By normalizing $|\alpha\rangle$ show that $c_0 = e^{-|\alpha|^2/2}$.

It follows from (b) and (c) that the probability to find the system in the state $|n\rangle$ is

$$P_n = |c_n|^2 = \frac{(|\alpha|^2)^n}{n!} e^{-|\alpha|^2},$$

which has the form of a Poisson distribution.

(d) Consider the limit of large quantum number n and, using Stirling's approximation, determine the most probable value of n (i.e., the value of n which maximizes P_n).

(e) Now consider the time evolution of this state in the Schrödinger picture. Show that the time-dependent state $|\alpha(t)\rangle$ remains an eigenstate of a but now with a time-dependent eigenvalue,

$$a|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle; \quad \alpha(t) = e^{-i\omega t}\alpha(0).$$

It follows that coherent states *stay* coherent states and so continue to have the minimal uncertainty product.

(f) Compute the commutator $[a, T(l)]$ where $T(l) = e^{-i\hat{p}l/\hbar}$ is the translation operator. Using your result show that $T(l)|0\rangle$ (i.e. the ground state translated through distance l) is an eigenstate of a (and hence a coherent state), and determine the corresponding eigenvalue.

7.4 Consider a one-dimensional Harmonic oscillator with Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2.$$

The system is prepared in an initial state at time $t = 0$ with position-space wave function

$$\psi(x, 0) = Ax^2 e^{-\frac{m\omega}{2\hbar}x^2}.$$

(a) Determine the normalization constant A.

(b) Express the normalized $\psi(x, 0)$ as a linear combination of Harmonic oscillator eigenstates.

(c) Working in the Schrödinger picture, find the time-dependent position-space wave function $\psi(x, t)$.

(d) Show that the expectation value of $\hat{x}$ in this state is zero at all times. Find the expectation value of $\hat{x}^2$ in this state as a function of time and hence determine $\langle(\Delta x)^2\rangle$. Can you interpret your result?

7.5 Schrödinger equation in momentum-space representation.

(a) Using the fact that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar},$$

show that

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle.$$

Now consider a one-dimensional harmonic oscillator with Hamiltonian,

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

(b) Obtain the momentum-space representation of the time-independent Schrödinger equation, $H|E\rangle = E|E\rangle$. This will be a differential equation for momentum-space wave function $\tilde{\psi}_E(p) = \langle p|E\rangle$.

(c) Based on the form of this equation, determine the momentum-space wave functions for the energy eigenstates. [Note: there is no need to solve this equation "from scratch." You should be able to write down the eigenfunctions by analogy with the position-space Schrödinger equation.]

7.6 Half-harmonic potential.

A one-dimensional quantum particle of mass m is subject to the "half-harmonic" potential

$$V(x) = \begin{cases} \frac{1}{2}kx^2 & \text{for } x > 0, \\ \infty & \text{for } x < 0. \end{cases}$$

(a) What is the ground state energy for this particle?

(b) Determine the normalized position-space wave function for the ground state.

Now assume the particle is in the ground state when the potential suddenly changes to a *full* harmonic potential, $V(x) = \frac{1}{2}kx^2$ for $-\infty < x < \infty$.

(c) What is the probability to find the particle in the ground state of the new potential?

(d) What is the probability to find the particle in the first-excited state of the new potential?