Physics 5645
Quantum Mechanics A
Problem Set VIII

Due: Thursday, Nov 7, 2019

8.1 Uncertainty estimates.

Estimate the zero-point energy for a particle of mass m in the following potentials

(a) $V(x) = \alpha x^4$ in one dimension where $\alpha > 0$.

(b) $V(r) = -e^2/r$ in three dimensions. This is the potential felt by an electron in a hydrogen atom. Express the estimate in eV, taking e and m to be the charge and mass of the electron.

8.2 Particle in a three-dimensional box.

Consider a three-dimensional quantum particle of mass m confined to cubic box of volume L^3. Choose the origin of your coordinate system to be one of the corners of the cube so that the potential is 0 in the region $0 < x < L$, $0 < y < L$, $0 < z < L$, and infinite everywhere else.

(a) Obtain the normalized energy eigenfunctions and corresponding energy eigenvalues for this particle.

(b) Determine the degeneracies of the first six energy eigenvalues.

8.3 Three-dimensional harmonic oscillator.

A three-dimensional quantum particle of mass m experiences the harmonic potential

$$V(x) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2).$$

(a) Show that the (in general degenerate) energy eigenvalues are

$$E_n = \left(n + \frac{3}{2}\right)\hbar\omega, \quad n = 0, 1, 2, 3, \cdots$$

(b) Write down the corresponding position-space eigenfunctions for this particle in terms of the one-dimensional Harmonic oscillator wave functions. Reexpress the first four states in spherical coordinates.

(c) Show that the degeneracy of the energy level $E_n = (n + 3/2)\hbar\omega$ is $(n + 1)(n + 2)/2$.

8.4 Infinitesimal Rotations.

Let $R(\hat{n}\phi)$ by the rotation matrix which determines how the components of a vector $\vec{v}$ transform under rotation through angle ϕ about axis $\hat{n}$. For rotations about the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ axes these matrices are,

$$R(\hat{\imath}\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \quad R(\hat{\jmath}\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \quad R(\hat{k}\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Verify that, for $|\epsilon_x|, |\epsilon_y| \ll 1$,

$$R(-\epsilon_y\hat{\jmath})R(-\epsilon_x\hat{\imath})R(\epsilon_y\hat{\jmath})R(\epsilon_x\hat{\imath}) = R(-\epsilon_x\epsilon_y\hat{k}) + \cdots,$$

where $\cdots$ corresponds to terms which are of order ϵ_x^2, ϵ_y^2 or higher.

Hint: To do this it is enough to expand the relevant R matrices to *first* order in ϵ_x and ϵ_y. Then, when multiplying these matrices out, you can drop any terms of order ϵ_x^2, ϵ_y^2 or higher.

Let $D(R(\hat{n}\phi))$ be the unitary operator which rotates quantum states about the axis $\hat{n}$ through the angle ϕ. For an infinitesimal rotation we have

$$D(R(\hat{n}d\phi)) = 1 - i\frac{\hat{n}\cdot\vec{J}}{\hbar}d\phi.$$

where $\vec{J} = (J_x, J_y, J_z)$ is the angular momentum operator.

(a) Verify that

$$D(R(-\epsilon_y\hat{\jmath}))D(R(-\epsilon_x\hat{\imath}))D(R(\epsilon_y\hat{\jmath}))D(R(\epsilon_x\hat{\imath})) = 1 + \frac{1}{\hbar^2}[J_x, J_y]\epsilon_x\epsilon_y + \cdots,$$

where, $\vec{J}$ is the vector angular momentum operator, and, again, $\cdots$ indicates terms which are of order ϵ_x^2 or ϵ_y^2 or higher.

By comparing your result to that of Part (a) above, deduce the fundamental angular momentum commutation relation,

$$[J_x, J_y] = i\hbar J_z.$$

8.5 Prove the identity

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b}\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\sigma,$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\mathbb{1}$ is the 2×2 identity matrix in the following two different ways.

(a) By using the fact that

$$\sigma_i\sigma_j = \delta_{ij}1 + i\sum_k \epsilon_{ijk}\sigma_k,$$

which you proved in Problem 3.2(a).

(b) By using the fact that any 2×2 matrix M can be expressed as

$$M = c\mathbb{1} + \vec{d}\cdot\sigma,$$

where

$$c = \frac{1}{2}Tr[M] \text{ and } d_i = \frac{1}{2}Tr[\sigma_i M],$$

which you proved in Problem 3.4(c), and also using

$$Tr[\sigma_i\sigma_j] = 2\delta_{ij}\mathbb{1}, \quad Tr[\sigma_i\sigma_j\sigma_k] = 2i\epsilon_{ijk}\mathbb{1}.$$

8.6 Let $|\psi\rangle_R$ be the state of a spin-1/2 particle obtained by applying the rotation operator for a z-axis rotation through angle ϕ to the state $|\psi\rangle$,

$$|\psi\rangle_R = e^{-iS_z\phi/\hbar}|\psi\rangle.$$

In class we showed that

$${}_R\langle\psi|S_x|\psi\rangle_R = \cos\phi\langle\psi|S_x|\psi\rangle - \sin\phi\langle\psi|S_y|\psi\rangle.$$

Show that

$${}_R\langle\psi|S_y|\psi\rangle_R = \sin\phi\langle\psi|S_x|\psi\rangle + \cos\phi\langle\psi|S_y|\psi\rangle,$$

and

$${}_R\langle\psi|S_z|\psi\rangle_R = \langle\psi|S_z|\psi\rangle.$$

That is, show that

$$\begin{pmatrix} {}_R\langle\psi|S_x|\psi\rangle_R \\ {}_R\langle\psi|S_y|\psi\rangle_R \\ {}_R\langle\psi|S_z|\psi\rangle_R \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \langle\psi|S_x|\psi\rangle \\ \langle\psi|S_y|\psi\rangle \\ \langle\psi|S_z|\psi\rangle \end{pmatrix},$$

so that $\langle\vec{S}\rangle$ transforms under rotations as a vector (see $R(\hat{k}\phi)$ in Problem 8.4).