Physics 5645
Quantum Mechanics A
Problem Set IX

Due: Thursday, Nov 14, 2019

9.1 Problem 3.3, Sakurai and Napolitano, Pg. 256.

9.2 Problem 3.15, Sakurai and Napolitano, Pg. 257.

9.3 Spin-1 rotation matrix.

Recall that the matrix representation of J_x for a spin-1 particle is,

$$J_x^{(1)} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(a) Compute the matrices ${J_x^{(1)}}^2$ and ${J_x^{(1)}}^3$. You should find that ${J_x^{(1)}}^3 = \hbar^2 J_x^{(1)}$.

(b) By Taylor expanding the exponential, and using the results of Part (a), show that the spin-1 matrix representation of the rotation operator for an x-axis rotation by angle ϕ, $D(R(\phi\hat{i})) = e^{-i\frac{J_x}{\hbar}\phi}$, is

$$\begin{aligned} e^{-i\frac{J_x^{(1)}}{\hbar}\phi} &= \mathbb{1} - i\sin\phi\left(\frac{J_x^{(1)}}{\hbar}\right) + (\cos\phi - 1)\left(\frac{J_x^{(1)}}{\hbar}\right)^2 \\ &= \begin{pmatrix} \frac{1}{2}(1+\cos\phi) & -\frac{i}{\sqrt{2}}\sin\phi & -\frac{1}{2}(1-\cos\phi) \\ -\frac{i}{\sqrt{2}}\sin\phi & \cos\phi & -\frac{i}{\sqrt{2}}\sin\phi \\ -\frac{1}{2}(1-\cos\phi) & -\frac{i}{\sqrt{2}}\sin\phi & \frac{1}{2}(1+\cos\phi) \end{pmatrix}. \end{aligned}$$

(c) Denote the spin-1 eigenstates of J_z by $|1m\rangle$ where $J_z|1m\rangle = m\hbar|1m\rangle$ with $m = -1, 0, 1$. Using your result from Part (b), express the state obtained by rotating $|11\rangle$ about the x axis through angle ϕ in the $\{|1m\rangle\}$ basis. Determine the probabilities to find $J_z = +\hbar$, 0, and $-\hbar$ in this state and obtain the expectation value of $\vec{J} = (J_x, J_y, J_z)$.

9.4 Spin-3/2 matrix representation of J_x.

(a) Using

$$J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}|j,m\pm 1\rangle, \quad \text{where} \quad J_\pm = J_x \pm iJ_y,$$

determine $J_x^{(3/2)}$, the 4×4 matrix representation of J_x for $j = 3/2$ in the J_z basis.

(b) Find the eigenvalues of $J_x^{(3/2)}$ and confirm that they are $-\frac{3}{2}\hbar, -\frac{1}{2}\hbar, \frac{1}{2}\hbar, \frac{3}{2}\hbar$.