## Physics 5646

## Quantum Mechanics B

Problem Set I
Due: Thursday, Jan 18, 2018
DEADLINE EXTENDED TO TUESDAY, JAN 23
1.1 Consider two spin- $1 / 2$ particles, with spin operators $\mathbf{S}_{1}$ and $\mathbf{S}_{2}$. Apply the operator $\mathbf{S}^{2}=\left(\mathbf{S}_{1}+\mathbf{S}_{2}\right)^{2}$ to the the three triplet states,

$$
|11\rangle=|++\rangle, \quad|10\rangle=\frac{1}{\sqrt{2}}(|+-\rangle+|-+\rangle), \quad|1-1\rangle=|--\rangle,
$$

and the singlet state,

$$
|00\rangle=\frac{1}{\sqrt{2}}(|+-\rangle-|-+\rangle),
$$

and verify that in all cases $\mathbf{S}^{2}|s m\rangle=s(s+1) \hbar^{2}|s m\rangle$.
Hint: To do this you may find the following identity useful:

$$
\mathbf{S}_{\mathbf{1}} \cdot \mathbf{S}_{\mathbf{2}}=S_{1 z} S_{2 z}+\frac{1}{2}\left(S_{1+} S_{2-}+S_{1-} S_{2+}\right)
$$

1.2 Calculate the Clebsch-Gordan coefficients for
(a) $1 \otimes \frac{1}{2}=\frac{1}{2} \oplus \frac{3}{2}$, and
(b) $1 \otimes 1=0 \oplus 1 \oplus 2$.

In doing this, you may use the property that

$$
\left\langle j_{1} m_{1}, j_{2} m_{2} \mid j, m\right\rangle=(-1)^{j_{1}+j_{2}-j}\left\langle j_{1}\left(-m_{1}\right), j_{2}\left(-m_{2}\right) \mid j(-m)\right\rangle
$$

to reduce the number of coefficients you need to explicitly calculate.
1.3 Argue that $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}=\frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$, and verify that the dimensionality of the Hilbert space is the same on both sides of this equation.
1.4 Obtain the Clebsch-Gordan coefficients for $j_{1} \otimes \frac{1}{2}=\left(j_{1}-\frac{1}{2}\right) \oplus\left(j_{1}+\frac{1}{2}\right)$ for arbitrary $j_{1} \neq 0$. You can do this by determining the coefficients $A$ and $B$ in

$$
|j m\rangle=A\left|j_{1}(m+1 / 2), \frac{1}{2}-\frac{1}{2}\right\rangle+B\left|j_{1}(m-1 / 2), \frac{1}{2} \frac{1}{2}\right\rangle
$$

for which $|j m\rangle$ is a normalized eigenstate of $\mathbf{J}^{2}=\left(\mathbf{J}_{1}+\mathbf{J}_{2}\right)^{2}$.
Answer: For $j=j_{1} \pm \frac{1}{2}$ and $m=-j, \cdots, j$,

$$
\begin{aligned}
A & =\left\langle j_{1}\left(m+\frac{1}{2}\right), \left.\frac{1}{2}-\frac{1}{2} \right\rvert\, j m\right\rangle=\sqrt{\frac{1}{2} \mp \frac{m}{2 j_{1}+1}}, \\
B & =\left\langle j_{1}\left(m-\frac{1}{2}\right), \left.\frac{1}{2} \frac{1}{2} \right\rvert\, j m\right\rangle= \pm \sqrt{\frac{1}{2} \pm \frac{m}{2 j_{1}+1}}
\end{aligned}
$$

where the phase is chosen to be consistent with our convention.
1.5 An electron in a hydrogen atom is in the state $|\psi\rangle$ with position/spin-space wave function (in spinor notation),

$$
\binom{\psi_{+}(\vec{r})}{\psi_{-}(\vec{r})}=R_{21}(r)\left[\sqrt{\frac{1}{3}} Y_{1}^{0}(\theta, \phi)\binom{1}{0}+\sqrt{\frac{2}{3}} Y_{1}^{1}(\theta, \phi)\binom{0}{1}\right]
$$

where $R_{21}(r)$ is the $n=2, l=1$ hydrogen atom radial function.
(a) What are the possible results of measuring the orbital angular momentum squared, $\mathbf{L}^{2}$, and what are the probabilities for each possibility?
(b) What are the possible results of measuring $z$-component of the orbital angular momentum, $L_{z}$, and what are the probabilities for each possibility?
(c) What are the possible results of measuring the spin angular momentum squared, $\mathbf{S}^{2}$, and what are the probabilities for each possibility?
(d) What are the possible results of measuring the $z$-component of the spin angular momentum, $S_{z}$, and what are the probabilities for each possibility?
(e) What are the possible results of measuring the total angular momentum squared, $\mathbf{J}^{2}=$ $(\mathbf{L}+\mathbf{S})^{2}$, and what are the probabilities for each possibility?
(f) What are the possible results of measuring the $z$-component of the total angular momentum, $J_{z}$, and what are the probabilities for each possibility?
(g) If you measure the position of the particle, what is the probability density to find it at the point $r, \theta, \phi$ ?
(h) If you measure the $z$-component of the spin and the distance from the origin, what is the probability density for finding the particle with spin up at a distance $r$ from the origin?

Hint: To do parts (e) and (f) you may find the Clebsch-Gordan coefficients computed in Problem 1.4 helpful.

