## Physics 5646 Quantum Mechanics B Problem Set I

## Due: Thursday, Jan 18, 2018 DEADLINE EXTENDED TO TUESDAY, JAN 23

1.1 Consider two spin-1/2 particles, with spin operators  $S_1$  and  $S_2$ . Apply the operator  $S^2 = (S_1 + S_2)^2$  to the three triplet states,

$$|11\rangle = |++\rangle, \quad |10\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), \quad |1 - 1\rangle = |--\rangle,$$

and the singlet state,

$$|00\rangle = \frac{1}{\sqrt{2}} \left(|+-\rangle - |-+\rangle\right),$$

and verify that in all cases  $\mathbf{S}^2 |sm\rangle = s(s+1)\hbar^2 |sm\rangle$ .

Hint: To do this you may find the following identity useful:

$$\mathbf{S_1} \cdot \mathbf{S_2} = S_{1z}S_{2z} + \frac{1}{2}\left(S_{1+}S_{2-} + S_{1-}S_{2+}\right).$$

1.2 Calculate the Clebsch-Gordan coefficients for

- (a)  $1 \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$ , and
- (b)  $1 \otimes 1 = 0 \oplus 1 \oplus 2$ .

In doing this, you may use the property that

$$\langle j_1 m_1, j_2 m_2 | j, m \rangle = (-1)^{j_1 + j_2 - j} \langle j_1(-m_1), j_2(-m_2) | j(-m) \rangle$$

to reduce the number of coefficients you need to explicitly calculate.

1.3 Argue that  $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$ , and verify that the dimensionality of the Hilbert space is the same on both sides of this equation.

1.4 Obtain the Clebsch-Gordan coefficients for  $j_1 \otimes \frac{1}{2} = (j_1 - \frac{1}{2}) \oplus (j_1 + \frac{1}{2})$  for arbitrary  $j_1 \neq 0$ . You can do this by determining the coefficients A and B in

$$|jm\rangle = A|j_1(m+1/2), \frac{1}{2} - \frac{1}{2}\rangle + B|j_1(m-1/2), \frac{1}{2} \frac{1}{2}\rangle,$$

for which  $|jm\rangle$  is a normalized eigenstate of  $\mathbf{J}^2 = (\mathbf{J}_1 + \mathbf{J}_2)^2$ . Answer: For  $j = j_1 \pm \frac{1}{2}$  and  $m = -j, \dots, j$ ,

$$A = \left\langle j_1\left(m + \frac{1}{2}\right), \frac{1}{2} - \frac{1}{2} \middle| jm \right\rangle = \sqrt{\frac{1}{2} \mp \frac{m}{2j_1 + 1}}, \\ B = \left\langle j_1\left(m - \frac{1}{2}\right), \frac{1}{2} \left| \frac{1}{2} \right| jm \right\rangle = \pm \sqrt{\frac{1}{2} \pm \frac{m}{2j_1 + 1}},$$

where the phase is chosen to be consistent with our convention.

1.5 An electron in a hydrogen atom is in the state  $|\psi\rangle$  with position/spin-space wave function (in spinor notation),

$$\begin{pmatrix} \psi_+(\vec{r}) \\ \psi_-(\vec{r}) \end{pmatrix} = R_{21}(r) \left[ \sqrt{\frac{1}{3}} Y_1^0(\theta,\phi) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sqrt{\frac{2}{3}} Y_1^1(\theta,\phi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right],$$

where  $R_{21}(r)$  is the n = 2, l = 1 hydrogen atom radial function.

- (a) What are the possible results of measuring the orbital angular momentum squared, L<sup>2</sup>, and what are the probabilities for each possibility?
- (b) What are the possible results of measuring z-component of the orbital angular momentum,  $L_z$ , and what are the probabilities for each possibility?
- (c) What are the possible results of measuring the spin angular momentum squared,  $S^2$ , and what are the probabilities for each possibility?
- (d) What are the possible results of measuring the z-component of the spin angular momentum,  $S_z$ , and what are the probabilities for each possibility?
- (e) What are the possible results of measuring the total angular momentum squared,  $\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2$ , and what are the probabilities for each possibility?
- (f) What are the possible results of measuring the z-component of the total angular momentum,  $J_z$ , and what are the probabilities for each possibility?
- (g) If you measure the position of the particle, what is the probability density to find it at the point  $r, \theta, \phi$ ?

(h) If you measure the z-component of the spin and the distance from the origin, what is the probability density for finding the particle with spin up at a distance r from the origin?

**Hint**: To do parts (e) and (f) you may find the Clebsch-Gordan coefficients computed in Problem 1.4 helpful.