Physics 5646 Quantum Mechanics B Problem Set X

Due: Thursday, Apr 19, 2018

10.1 Validity of Born approximation.

Consider scattering of a particle of mass m and energy $E = \hbar^2 k^2 / (2m)$ from the potential $V(r) = -V_0 e^{-r/a}$.

(a) Compute the scattering amplitude, $f(\theta)$, and the differential cross section, $d\sigma/d\Omega$, within the first-order Born approximation.

Recall the exact integral equation for $\psi(\vec{r})$ (Lippmann-Schwinger equation) which we used to develop the Born approximation:

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{ikz} - \frac{1}{4\pi} \frac{2m}{\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r'}|}}{|\vec{r}-\vec{r'}|} V(\vec{r'})\psi(\vec{r'})d^3r'.$$
 (1)

- (b) Calculate the value of $\psi(0)$ in the first-order Born approximation. To do this, approximate the $\psi(\vec{r'})$ appearing in the integrand on the right-hand side of (1) by the zeroth-order state $\frac{1}{(2\pi)^{3/2}}e^{ikz'}$ and perform the integral for $V(r) = -V_0e^{-r/a}$.
- (c) Under what condition is the deviation of $\psi(0)$ from its zeroth-order value $\frac{1}{(2\pi)^{3/2}}e^{ikz}\Big|_{\vec{r}=0} = \frac{1}{(2\pi)^{3/2}}$ a small correction? Argue that when this condition is met, the first-order Born approximation should be valid. Does the Born approximation get better or worse with increasing energy?

10.2 Consider a beam of particle of mass m and energy E scattering from a hard sphere of radius a (i.e., a potential for which $V = \infty$ for r < a and V = 0 for r > a). Use two partial waves (l = 0 and l = 1) to show that for small ka, where $k = \sqrt{2mE/\hbar^2}$, the differential cross section is

$$\frac{d\sigma}{d\Omega} \simeq a^2 [1 + 2(ka)^2 \cos\theta],$$

and the total cross section is

$$\sigma \simeq 4\pi a^2 [1 + (ka)^4/3].$$

10.3 Consider the scattering of a particle of mass m with energy E from the 'soft-sphere' potential

$$V(\vec{r}) = \begin{cases} V_0, & \text{if } r \leq a, \\ 0, & \text{if } r > a, \end{cases}$$

using partial wave analysis. Assume that $E < V_0$.

(a) Show that the exact l = 0 phase shift for scattering from this potential is given by

$$\delta_0 = -ka + \tan^{-1}\left(\frac{ka}{\rho a} \tanh \rho a\right),$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$
 and $\rho = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}.$

- (b) Find $f(\theta)$, $d\sigma/d\Omega$, and σ for soft-sphere scattering in the $ka \ll 1$ limit.
- (c) Expand your results to Part (c) to lowest order in V_0 and show that they agree with the results of the low-energy Born approximation for this potential.
- (d) Show that in the limit $V_0 \to \infty$ your answers to Part (b) agree with the results for hard sphere scattering in the low energy limit.

10.4 Consider the scattering of a particle of mass m with energy E from a delta-function shell of radius a,

$$V(\vec{r}) = \alpha \delta(r-a)$$

(a) Show that the exact l = 0 phase shift for scattering from this potential is given by

$$\tan \delta_0 = \frac{-\frac{2m\alpha}{\hbar^2 k} \sin^2(ka)}{1 + \frac{2m\alpha}{\hbar^2 k} \sin(ka) \cos(ka)},$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}.$$

(b) Find $f(\theta)$, $d\sigma/d\Omega$, and σ for scattering from this potential in the $ka \ll 1$ limit.

- (c) Expand your result for $f(\theta)$ to second order in α and show that what you find agrees with the results of the low-energy first- and second-order Born approximations for this potential you obtained in Problem 9.4.
- (d) Show that in the limit $\alpha \to \infty$ your answers to Part (b) agree with the results for hard sphere scattering in the low energy limit.