Physics 5646 Quantum Mechanics B Problem Set VII

Due: Tuesday, Mar 20, 2018

7.1 Projection Theorem.

In what follows you may use the fact that

$$\langle 10, jj | jj \rangle = -\sqrt{\frac{j}{j+1}},$$

(this is a special case of one of the Clebsch-Gordan coefficients I calculated in class) and the Wigner-Eckart theorem, which, using Sakurai's convention for the reduced matrix element, is

$$\langle \alpha', j'm' | T_q^{(k)} | \alpha, jm \rangle = \langle kq, jm | j'm' \rangle \frac{\langle \alpha', j' | | T^{(k)} | | \alpha, j \rangle}{\sqrt{2j+1}}.$$

(a) Show that

$$\frac{\langle \alpha', j' || J^{(1)} || \alpha, j \rangle}{\sqrt{2j+1}} = -\delta_{\alpha \alpha'} \delta_{jj'} \sqrt{j(j+1)}.$$

(b) Consider an arbitrary vector operator $\vec{V} = (V_x, V_y, V_z)$. Using the fact that

$$\vec{J} \cdot \vec{V} = J_z V_z + \frac{1}{2} \left(J_+ V_- + J_- V_+ \right),$$

where $V_{\pm} = V_x \pm i V_y$, show that

$$\langle \alpha', jm' | \vec{J} \cdot \vec{V} | \alpha, jm \rangle = c_{jm} \frac{\langle \alpha', j| |V^{(1)}| | \alpha, j \rangle}{\sqrt{2j+1}}$$

where c_{jm} does not depend on α , α' or \vec{V} . Then use the result of Part (a) to show that

$$c_{jm} = -\hbar\sqrt{j(j+1)}\delta_{mm'}.$$

(c) Using the previous result, show that

$$\langle \alpha', jm' | V_q^{(1)} | \alpha, jm \rangle = \frac{\langle \alpha', jm | \vec{J} \cdot \vec{V} | \alpha, jm \rangle}{\hbar^2 j (j+1)} \langle jm' | J_q^{(1)} | jm \rangle$$

This result is known as the projection theorem.

(d) Use the projection theorem to show that if $|njm; ls\rangle$ is the H-atom state with total angular momentum quantum number j, J_z quantum number m, and orbital and spin quantum numbers l and s (= 1/2), that

$$\langle njm; ls|V_Z|njm; ls \rangle = -\frac{e\hbar}{2m_e c} g_{jls} mB; \quad g_{jls} = 1 + \frac{j(j+1) - l(l+1) + 3/4}{2j(j+1)},$$

where

$$V_z = -\frac{e}{2m_ec}(L_z + 2S_z)B,$$

is the Zeeman term describing the coupling of a magnetic field $\vec{B} = B\hat{k}$ to the orbital and spin magnetic moments of the electron.

7.2 Hyperfine Structure.

In addition to the Coulomb interaction, the electron and proton interact via the so-called hyperfine interaction. In the 1s state of the Hydrogen atom, this interaction takes the form

$$H_{hf} = A\vec{S}_e \cdot \vec{S}_p,$$

where \vec{S}_e and \vec{S}_p are the electron and proton spins, respectively.

(a) Show that H_{hf} splits the ground state of the hydrogen atom into two levels with energies,

$$E_{+} = E_{1} + \frac{\hbar^{2}A}{4}, \qquad E_{-} = E_{1} - \frac{3\hbar^{2}A}{4},$$

where the electron and proton spins form a singlet (triplet) in the state(s) with energy E_{-} (E_{+}). Here E_{1} is the ground state energy of hydrogen without H_{hf} ($E_{1} = -\frac{e^{2}}{2a_{0}}$ in the absence of fine structure corrections).

(b) The origin of the hyperfine coupling is the interaction between the magnetic dipole moments of the electron, $\vec{\mu}_e = -\frac{|e|}{m_e c} \vec{S}_e$, and proton, $\vec{\mu}_p = g_p \frac{|e|}{2m_p c} \vec{S}_p$, where m_e and m_p are electron and proton masses, respectively, and $g_p \simeq 5.7$ is the proton g-factor. While this interaction is somewhat complicated, the net effect in the 1s state is an effective interaction of the form

$$H \simeq -\frac{1}{a_0^3} \vec{\mu}_e \cdot \vec{\mu}_p$$

Using this, estimate the coupling A in Part (a), and show that the splitting between E_+ and E_- is on the order of

$$E_+ - E_- \sim \frac{m_e}{m_p} \alpha^2 \frac{e^2}{2a_0}$$

Thus we see that the hyperfine splitting, like the fine structure splittings, is on the order of $\alpha^2 E_1$, but is also smaller by an additional factor of m_e/m_p . Using this result, estimate the order of magnitude of the wavelength of a photon emitted when the atom undergoes a transition from E_+ to E_- . [The actual wavelength for this transition is 21 cm — the famous "21-cm line" of great importance in radio astronomy.]

7.3 He Atom: First-Order Ground State Energy Shift.

The Hamiltonian for a He-like atom in which two electrons are bound to a nucleus of charge Z|e| is

$$H = H_0 + H_{12},$$

where

$$H_0 = \frac{\hat{\vec{p}_1}^2}{2m} - \frac{Ze^2}{\hat{r}_1} + \frac{\hat{\vec{p}_2}^2}{2m} - \frac{Ze^2}{\hat{r}_2} \quad \text{and} \quad H_{12} = \frac{e^2}{|\hat{\vec{r}_1} - \hat{\vec{r}_2}|}.$$

Let H_0 , which describes two noninteracting electrons in the presence of the charge +Z|e|nucleus, be the unperturbed Hamiltonian, and let $|\psi_{GS}^0\rangle$ denote the unperturbed ground state of H_0 , which, in the position representation, is

$$\langle \vec{r_1}, \vec{r_2}, m_{s1}, m_{s2} | \psi_{GS}^0 \rangle = \psi_{100,Z}(\vec{r_1}) \psi_{100,Z}(\vec{r_2}) \chi_{singlet}(m_{s1}, m_{s2}),$$

where

$$\psi_{100,Z}(\vec{r}) = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} e^{-Zr/a_0}.$$

Calculate that the first order ground state energy shift due to the perturbation H_{12} ,

$$E_{GS}^{1} = \langle \psi_{GS}^{0} | H_{12} | \psi_{GS}^{0} \rangle = \int |\psi_{100,Z}(\vec{r_1})|^2 |\psi_{100,Z}(\vec{r_2})|^2 \frac{e^2}{|\vec{r_1} - \vec{r_2}|} d^3 r_1 d^3 r_2.$$

[Answer: $E_{GS}^1 = \frac{5}{8} \frac{Ze^2}{a_0}$.]

7.4 Use the following trial wave function with variational parameter $\lambda > 0$,

$$\psi_{\lambda}(x) = \begin{cases} A_{\lambda}(x^2 - \lambda^2)^2, & -\lambda < x < \lambda \\ 0, & \text{otherwise} \end{cases}$$

to estimate the ground state energy of the one-dimensional harmonic oscillator with Hamiltonian $H = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2$. Compare your result to the exact ground state energy $\frac{1}{2}\hbar\omega$.

7.5 Use the variational wave function $\psi_{\alpha}(\vec{r}) = A_{\alpha}e^{-\alpha r^2}$ to estimate the ground state energy of the hydrogen atom with Hamiltonian $H = \frac{\hat{p}^2}{2m} - \frac{e^2}{\hat{r}}$. Compare your variational result to the exact result.