Physics 5646
Quantum Mechanics B
Problem Set X

Due: Friday, Apr 19, 2019

10.1 Validity of Born approximation.

Consider scattering of a particle of mass m and energy $E = \hbar^2 k^2/(2m)$ from the potential $V(r) = -V_0 e^{-r/a}$.

(a) Compute the scattering amplitude, $f(\theta)$, and the differential cross section, $d\sigma/d\Omega$, within the first-order Born approximation.

Recall the exact integral equation for $\psi(\vec{r})$ (Lippmann-Schwinger equation) which we used to develop the Born approximation:

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}} e^{ikz} - \frac{1}{4\pi}\frac{2m}{\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} V(\vec{r}')\psi(\vec{r}') d^3 r'. \tag{1}$$

(b) Calculate the value of $\psi(0)$ in the first-order Born approximation. To do this, approximate the $\psi(\vec{r}')$ appearing in the integrand on the right-hand side of (1) by the zeroth-order state $\frac{1}{(2\pi)^{3/2}} e^{ikz'}$ and perform the integral for $V(r) = -V_0 e^{-r/a}$.

(c) Under what condition is the deviation of $\psi(0)$ from its zeroth-order value $\left.\frac{1}{(2\pi)^{3/2}} e^{ikz}\right|_{\vec{r}=0} = \frac{1}{(2\pi)^{3/2}}$ a small correction? Argue that when this condition is met, the first-order Born approximation should be valid. Does the Born approximation get better or worse with increasing energy?

10.2 Consider a beam of particle of mass m and energy E scattering from a hard sphere of radius a (i.e., a potential for which $V = \infty$ for $r < a$ and $V = 0$ for $r > a$). Use two partial waves ($l = 0$ and $l = 1$) to show that for small ka, where $k = \sqrt{2mE/\hbar^2}$, the differential cross section is

$$\frac{d\sigma}{d\Omega} \simeq a^2[1 + 2(ka)^2\cos\theta],$$

and the total cross section is

$$\sigma \simeq 4\pi a^2[1 + (ka)^4/3].$$

10.3 Consider the scattering of a particle of mass m with energy E from the 'soft-sphere' potential

$$V(\vec{r}) = \begin{cases} V_0, & \text{if } r \leq a, \\ 0, & \text{if } r > a, \end{cases}$$

using partial wave analysis. Assume that $E < V_0$.

(a) Show that the *exact* $l = 0$ phase shift for scattering from this potential is given by

$$\delta_0 = -ka + \tan^{-1}\left(\frac{ka}{\rho a} \tanh \rho a\right),$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \text{and} \qquad \rho = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}.$$

(b) Find $f(\theta)$, $d\sigma/d\Omega$, and σ for soft-sphere scattering in the $ka \ll 1$ limit.

(c) Expand your results to Part (c) to lowest order in V_0 and show that they agree with the results of the low-energy Born approximation for this potential.

(d) Show that in the limit $V_0 \to \infty$ your answers to Part (b) agree with the results for hard sphere scattering in the low energy limit.

10.4 Consider the scattering of a particle of mass m with energy E from a delta-function shell of radius a,

$$V(\vec{r}) = \alpha\delta(r - a)$$

(a) Show that the *exact* $l = 0$ phase shift for scattering from this potential is given by

$$\tan \delta_0 = \frac{-\frac{2m\alpha}{\hbar^2 k} \sin^2(ka)}{1 + \frac{2m\alpha}{\hbar^2 k} \sin(ka)\cos(ka)},$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}.$$

(b) Find $f(\theta)$, $d\sigma/d\Omega$, and σ for scattering from this potential in the $ka \ll 1$ limit.

(c) Expand your result for $f(\theta)$ to second order in α and show that what you find agrees with the results of the low-energy first- and second-order Born approximations for this potential you obtained in Problem 9.4.

(d) Show that in the limit $\alpha \to \infty$ your answers to Part (b) agree with the results for hard sphere scattering in the low energy limit.